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Gauge-Fermion unification and flavour symmetry

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Abstract: After we study the 6-dimensional $\mathcal{N} = (1,1)$ supersymmetry breaking and R symmetry breaking on $M^4 \times T^2/Z_n$, we construct two $\mathcal{N} = (1,1)$ supersymmetric E_6 models on $M^4 \times T^2/Z_3$ where E_6 is broken down to $\mathrm{SO}(10) \times \mathrm{U}(1)_X$ by orbifold projection. In Model I, three families of the Standard Model fermions arise from the zero modes of bulk vector multiplet, and the R symmetry $\mathrm{U}(1)_F^I \times \mathrm{SU}(2)_{\mathbf{4}_-}$ can be considered as flavour symmetry. This may explain why there are three families of fermions in the nature. In Model II, the first two families come from the zero modes of bulk vector multiplet, and the flavour symmetry is similar. In these models, the anomalies can be cancelled, and we have very good fits to the SM fermion masses and mixings. We also comment on the $\mathcal{N} = (1,1)$ supersymmetric E_6 models on $M^4 \times T^2/Z_4$ and $M^4 \times T^2/Z_6$, SU(9) models on $M^4 \times T^2/Z_3$, and SU(8) models on T^2 orbifolds.

Keywords: Field Theories in Higher Dimensions, Supersymmetry Breaking, GUT.

Contents

1. Introduction

Supersymmetric Grand Unified Theory (SUSY GUT) is one of the most promising candidates which describe the fundamental interactions except gravity in the nature. Supersymmetry is an elegant solution to the gauge hierarchy problem, and the Grand Unified Theory gives us a simple understanding of the quantum numbers of quarks and leptons. Moreover, the success of gauge coupling unification in the Minimal Supersymmetric Standard Model strongly supports this idea. The electroweak symmetry can be broken by radiative corrections due to the large top quark Yukawa coupling, and the very tiny neutrino masses can be realized naturally by see-saw mechanism in the SUSY GUT.

Among several candidates of the GUT gauge groups such as SU(5), SO(10), and E_6, SO(10) has several particularly attractive features. SO(10) is the smallest semi-simple GUT group which does not require a particular particle content to cancel the gauge anomaly. And there exists the fermion unification, i.e., all the quarks and leptons including the right-handed neutrinos in each generation are unified to a single **16** dimensional spinor representation field, and then, we have the Yukawa unification. In addition, the doublet-triplet splitting can be explained by the Dimopoulos-Wilczek mechanism [1].

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However, it is very difficult to understand the fermion masses and mixings in SO(10) model due to the fermion unification. The Yukawa matrices have hierarchical structures and they are quite different among the quark and lepton sectors. On the one hand, all the mixing angles in the quark sector are small and the masses of quarks have hierarchical structures. The magnitude of the mass hierarchy is enormous, for instance, the ratio of the up-quark mass m_u to the top-quark mass m_t is approximately 10^{-5}. On the other hand, there exist the bilarge mixings in neutrino-lepton sector from solar neutrino, atmospheric neutrino and reactor neutrino experiments [2, 3, 4]. θ_{12}, which is the mixing angle for solar oscillation, is large with best fit around 32.6^o. And θ_{23}, which is the mixing angle for atmospheric oscillation, is also large with best fit around 45^o [5, 6]. However, the mixing angle θ_{13} is small, $\sin^2\theta_{13} \leq 0.04$ from the CHOOZ experiment [7].

How to understand the fermion masses and mixings in SO(10) model is an interesting question. The Froggatt-Nielsen (FN) mechanism is an elegant scenario to generate the hierarchical structures in Yukawa matrices [8]. Various flavour symmetries have been introduced to explain the fermion masses and mixings in the supersymmetric SO(10) models, for example, U(1) and extra discrete flavour symmetry [9, 10], U(2) flavour symmetry [11], $SU(2) \times U(1)$ flavour symmetry [12, 13], $SU(2) \times Z_2 \times Z_2 \times Z_2$ flavour symmetry [14], $SU(3) \times U(1)$ flavour symmetry [15], SU(3) flavour symmetry [16], etc. And the fermion-mass hierarchy in 5-dimensional SO(10) models can be realized by different fermions with different 5-dimensional wave-function profiles. Unlike the Froggatt-Nielsen mechanism, the SO(10) breaking effect may directly contribute to the source of the fermion mass hierarchy [17]. Moreover, if we introduce extra **10**-dimensional matter multiplets and mix or flip the d^c and doublet L in the **16** and those in the **10**-dimensional matter multiplets by the SO(10) breaking effect, we can explain the fermion masses and mixings by introducing U(1) flavour symmetry [18, 19]. This approach can be generalized to the 5-dimensional SO(10) models [20], and to the 6-dimensional SO(10) models where the Yukawa couplings can also be suppressed by the volume suppression factors [21]. However, this approach loses the fermion unification. By the way, the fermion masses and mixings in the SO(10) models with various flavour symmetries has been reviewed in ref. [22].

In addition, a new kind of scenario for the GUT breakings has been discussed extensively during the last several years [23]–[28]. The key point is that the supersymmetric GUT models exist in 5 or higher dimensions and are broken down to the 4-dimensional $\mathcal{N} = 1$ supersymmetric Standard like Models for the zero modes due to the discrete symmetries on the extra space manifolds, which become non-trivial constraints on the multiplets and gauge generators in the GUTs [26]. The attractive models have been constructed explicitly, where the supersymmetric 5-dimensional and 6-dimensional GUT models are broken down to the 4-dimensional $\mathcal{N} = 1$ supersymmetric $SU(3) \times SU(2) \times U(1)^{n-3}$ models, where n is the rank of GUT group, through the compactification on various orbifolds. The GUT gauge symmetry breaking and doublet-triplet splitting problems have been solved neatly by the discrete symmetry projections. And the partial gauge-fermion unification and the family symmetry as a remnant of the higher-dimensional R symmetry have been discussed in the 6-dimensional $\mathcal{N} = (1,1)$ supersymmetric SO(10) model on the space-time $M^4 \times T^2/Z_3$

where M^4 is the 4-dimensional Minkowski space-time [27]. Other interesting phenomenology, like μ problem, gauge coupling unification, non-supersymmetric GUT, gauge-Higgs unification, gauge-Yukawa unification, proton decay, etc, have also been studied [23]–[28].

Before we present our results, let us ask three interesting questions

(1) The maximal supersymmetry in the supersymmetric gauge theory is the 4-dimensional $\mathcal{N}=4$ supersymmetry, why is not the 4-dimensional $\mathcal{N}=4$ supersymmetry in the nature?

(2) Why are there three families of fermions in the Standard Model (SM) [29]?

(3) If SO(10) model is correct, what is the origin of the flavour symmetry which is used to explain the fermion masses and mixings?

In this paper, we first briefly review the T^2/Z_n orbifolds. And then, we consider the 6-dimensional $\mathcal{N}=(1,1)$ supersymmetric gauge theory on the space-time $M^4 \times T^2$, which corresponds to the 4-dimensional $\mathcal{N}=4$ supersymmetry, the maximal supersymmetry for gauge theory. The R symmetry in this case is $\mathrm{SO}(2)_{56} \times \mathrm{SU}(2)_{\mathbf{4}_+} \times \mathrm{SU}(2)_{\mathbf{4}_-}$, and the transformation properties of vector multiplet under this R symmetry are given. In order to preserve only the 4-dimensional $\mathcal{N}=1$ supersymmetry, we consider the supersymmetry breaking and R symmetry breaking on $M^4 \times T^2/Z_n$. We find that for T^2/Z_3 orbifold, the R symmetry is $\mathrm{SO}(2)_{56} \times \mathrm{U}(1)_{\mathbf{4}_+} \times \mathrm{SU}(2)_{\mathbf{4}_-}$. And for T^2/Z_4 and T^2/Z_6 orbifolds, the R symmetry is $\mathrm{SO}(2)_{56} \times \mathrm{U}(1)_{\mathbf{4}_+} \times \mathrm{U}(1)_{\mathbf{4}_-}$. We also present the transformation properties of the vector multiplet under these R symmetries. In addition, we explain how to embed the R symmetry into the $\mathrm{SU}(4)_R$ R symmetry, which is the maximal R symmetry for the 4-dimensional $\mathcal{N}=4$ supersymmetry. This embedding gives us an elegant way to understand the supersymmetry and R symmetry breakings.

In the 6-dimensional $\mathcal{N}=(1,1)$ supersymmetric gauge theory, there is a vector multiplet, which in the 4-dimensional $\mathcal{N}=1$ supersymmetry language corresponds to one gauge supermultiplet and three chiral multiplets in the adjoint representation. The Standard Model fermions may come from three chiral multiplets due to the orbifold gauge symmetry breaking, which may explain why there are three families of fermions in the Standard Model. And the R symmetry can give us the flavour symmetry to explain the fermion masses and mixings. To be concrete, we discuss the $\mathcal{N}=(1,1)$ supersymmetric E_6 models on the space-time $M^4 \times T^2/Z_3$ where the E_6 gauge symmetry is broken down to $\mathrm{SO}(10) \times \mathrm{U}(1)_X$ by orbifold projection. There are three SO(10) spinor representations **16** from the zero modes of bulk vector multiplet, which can be considered as three families of fermions. We study two models in detail. In Model I, three families of the Standard Model fermions arise from the zero modes of bulk vector multiplet, and the R symmetry $\mathrm{U}(1)_F^I \times \mathrm{SU}(2)_{\mathbf{4}_-}$ can be considered as flavour symmetry. Moreover, we show that the anomalies can be cancelled, and the very good fits to the Standard Model fermion (quark, lepton, and neutrino) masses and mixings can be obtained. In this model, we can understand why there are three families of the SM fermions in the nature and the origin of flavour symmetry. However, the $\mathrm{U}(1)_R^I$ R symmetry is not the traditional R symmetry in the low energy supersymmetry phenomenology and is broken by the Higgs VEVs. Because

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there is no unbroken Z_2 symmetry under which the ordinary particles are even while their superpartners are odd, there are no cold dark matter candidates (for example neutralino) from the superpartners. We point out that this $\mathrm{U}(1)_R^I$ R symmetry may be the unbroken $\mathrm{U}(1)_R$ R symmetry in the compactification of the weakly coupled heterotic string theory on Calabi-Yau manifolds. So, it is interesting to study its low energy phenomenology. In Model II, the first two families of the Standard Model fermions arise from the zero modes of bulk vector multiplet, and the R symmetry $\mathrm{U}(1)_F^{II} \times \mathrm{SU}(2)_{\mathbf{4}_-}$ can be considered as flavour symmetry. We also discuss the anomaly cancellations, and the fermion masses and mixings. Similar to those in Model I, we have very good fits to the Standard Model fermion masses and mixings. Unlike that in Model I, the $\mathrm{U}(1)_R^{II}$ R symmetry is the traditional $\mathrm{U}(1)_R$ R symmetry in the low energy supersymmetry phenomenology. However, we can not explain why there are three families of fermions in the nature.

Furthermore, we comment on the $\mathcal{N}=(1,1)$ supersymmetric E_6 models on $M^4 \times T^2/Z_4$ and $M^4 \times T^2/Z_6$ where there are two families of the Standard Model fermions from the zero modes of bulk vector multiplet. In these models, the $\mathrm{U}(1)_R^{III}$ R symmetry is the traditional $\mathrm{U}(1)_R$ R symmetry in the low energy supersymmetry phenomenology, and the $\mathrm{U}(1)_F^{III} \times \mathrm{U}(1)_{RF}$ R symmetry can be considered as flavour symmetry. In addition, we comment on the $\mathcal{N}=(1,1)$ supersymmetric SU(9) and SU(8) models on $M^4 \times T^2/Z_3$ where there are three or two families of the SM fermions from the zero modes of bulk vector multiplet, and comment on the SU(8) models on $M^4 \times T^2/Z_4$ and $M^4 \times T^2/Z_6$. The R symmetry and flavour symmetry are similar to those in E_6 models on the corresponding T^2 orbifolds.

This paper is organized as follows: in section 2, we briefly review the T^2 orbifolds, explain the 6-dimensional $\mathcal{N}=(1,1)$ supersymmetric gauge theory, and consider the supersymmetry breaking and R symmetry breaking by orbifold projection. In section 3, we discuss the $\mathcal{N}=(1,1)$ supersymmetric E_6 models on the space-time $M^4 \times T^2/Z_3$. We comment on the $\mathcal{N}=(1,1)$ supersymmetric E_6 models on $M^4 \times T^2/Z_4$ and $M^4 \times T^2/Z_6$ in section 4. We also comment on the $\mathcal{N}=(1,1)$ supersymmetric SU(9) models on $M^4 \times T^2/Z_3$ and SU(8) models on T^2 orbifolds in section 5. The discussions and conclusions are given in section 6.

2. 6-dimensional $\mathcal{N}=(1,1)$ supersymmetric gauge theory on $M^4 \times T^2/Z_n$

2.1 Discrete symmetry on T^2 and T^2 orbifolds

We consider the 6-dimensional space-time which can be factorized into a product of the ordinary 4-dimensional Minkowski space-time M^4, and the torus T^2 which is homeomorphic to $S^1 \times S^1$. The corresponding coordinates for the space-time are x^μ, $(\mu = 0,1,2,3)$, x^5 and x^6. The radii for the circles along x^5 direction and x^6 direction are R_1 and R_2, respectively. We define the complex coordinate

$$z \equiv \frac{1}{2}\left(x^5 + i x^6\right). \tag{2.1}$$

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In the complex coordinate, the torus T^2 can be defined by C^1 moduloed the equivalent classes:

$$z \sim z + \pi R_1 \,, \qquad z \sim z + \pi R_2 e^{\mathrm{i}\theta} \,. \tag{2.2}$$

The complete discrete symmetries on the torus are Z_2, Z_3, Z_4 and Z_6 [24]. And for Z_3, Z_4 and Z_6 discrete symmetries, we have to choose $R_1 = R_2 = R$.

For Z_3 symmetry, $\theta = 2\pi/3$. And the equivalent class $z \sim z + \pi R e^{\mathrm{i}2\pi/3}$ is equivalent to the equivalent class $z \sim z + \pi R e^{\mathrm{i}\pi/3}$. This torus admits the Z_3 symmetry generated by

$$\gamma_3 : \quad z \rightarrow e^{\mathrm{i}2\pi/3} z \,. \tag{2.3}$$

There are three fixed points: $z = 0$, $z = \pi R e^{\mathrm{i}\pi/6}/\sqrt{3}$, and $z = 2\pi R e^{\mathrm{i}\pi/6}/\sqrt{3}$. The 3-branes can be located at the fixed points where the extra particles can be put on.

For Z_4 symmetry, $\theta = \pi/2$. The Z_4 symmetry is generated by

$$\gamma_4 : \quad z \rightarrow e^{\mathrm{i}\pi/2} z \,. \tag{2.4}$$

There are two Z_4 fixed points: $z = 0$ and $z = \sqrt{2}\pi R e^{\mathrm{i}\pi/4}/2$, and two Z_2 fixed points: $z = \pi R/2$ and $z = \pi R e^{\mathrm{i}\pi/2}/2$.

For Z_6 symmetry, $\theta = \pi/3$. The Z_6 symmetry is generated by

$$\gamma_4 : \quad z \rightarrow e^{\mathrm{i}\pi/3} z \,. \tag{2.5}$$

There is one Z_6 fixed point $z = 0$, two Z_3 fixed points: $z = \pi R e^{\mathrm{i}\pi/6}/\sqrt{3}$ and $z = 2\pi R e^{\mathrm{i}\pi/6}/\sqrt{3}$, and three Z_2 fixed points: $z = \sqrt{3}\pi R e^{\mathrm{i}\pi/6}/2$, $z = \pi R/2$ and $z = \pi R e^{\mathrm{i}\pi/3}/2$.

In short, the T^2/Z_n orbifolds are obtained from T^2 moduloed the discrete symmetry Z_n. The KK mode expansions have been discussed in ref. [24].

2.2 6-dimensional $\mathcal{N} = (1,1)$ supersymmetric gauge theory on $M^4 \times T^2$

As we know, the 6-dimensional $\mathcal{N} = 1$ ((0, 1) or (1, 0)) supersymmetric gauge theory is chiral, where the gaugino (and gravitino) and the matters (hypermultiplets) have opposite chiralities, so, it has anomalies which are usually difficult to be cancelled by introducing the hypermultiplets. Therefore, we consider the 6-dimensional $\mathcal{N} = (1,1)$ supersymmetry, where the irreducible box anomalies and reducible anomalies vanish, and there is no global anomaly.

For the 6-dimensional $\mathcal{N} = (1,1)$ supersymmetry, there are four supercharges $\mathcal{Q}^{(6)}_{\mathbf{4}_+,A}$ and $\mathcal{Q}^{(6)}_{\mathbf{4}_-,B}$ with $A, B = 1,2$, which belong to the $\mathbf{4}_+$ and $\mathbf{4}_-$ spinor representations of the $\mathrm{SO}(5,1)$ with positive and negative chiralities, respectively [30, 31]. The pseudo-Majorana condition is imposed on these four supercharges,

$$\mathcal{Q}^{(6)}_{\mathbf{4}_+,A} = J_{AA'} (\mathcal{Q}^{(6)}_{\mathbf{4}_+,A'})^c \,, \qquad \mathcal{Q}^{(6)}_{\mathbf{4}_-,B} = J_{BB'} (\mathcal{Q}^{(6)}_{\mathbf{4}_-,B'})^c \,, \tag{2.6}$$

where c is the charge conjugation, and J is a symplectic invariant metric

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \tag{2.7}$$

Thus, only two supercharges are independent. The R symmetry for the 6-dimensional $\mathcal{N} = (1,1)$ supersymmetry is $\mathrm{SU}(2)_{\mathbf{4}_+} \times \mathrm{SU}(2)_{\mathbf{4}_-}$. The supercharges $\mathcal{Q}^{(6)}_{\mathbf{4}_+,A}$ and $\mathcal{Q}^{(6)}_{\mathbf{4}_-,B}$ form the fundamental representations under the R symmetry $\mathrm{SU}(2)_{\mathbf{4}_+}$ and $\mathrm{SU}(2)_{\mathbf{4}_-}$, respectively [30, 31].

For the gamma matrices and spinors in 4-dimension and 6-dimension, we follow the convention in ref. [31]. After the compactification on torus, each 6-dimensional supercharge is decomposed into two 4-dimensional Weyl spinors

$$\mathcal{Q}^{(6)}_{\mathbf{4}_+,1} = \begin{pmatrix} \mathcal{Q}^{(4)}_{\alpha,1} & 0 \\ 0 & \bar{\mathcal{Q}}^{\dot{\alpha},2}_{(4)} \end{pmatrix}, \qquad \mathcal{Q}^{(6)}_{\mathbf{4}_+,2} = \begin{pmatrix} \mathcal{Q}^{(4)}_{\alpha,2} & 0 \\ 0 & -\bar{\mathcal{Q}}^{\dot{\alpha},1}_{(4)} \end{pmatrix}, \tag{2.8}$$

$$\mathcal{Q}^{(6)}_{\mathbf{4}_-,1} = \begin{pmatrix} 0 & \mathcal{Q}^{(4)}_{\alpha,3} \\ \bar{\mathcal{Q}}^{\dot{\alpha},4}_{(4)} & 0 \end{pmatrix}, \qquad \mathcal{Q}^{(6)}_{\mathbf{4}_-,2} = \begin{pmatrix} 0 & \mathcal{Q}^{(4)}_{\alpha,4} \\ -\bar{\mathcal{Q}}^{\dot{\alpha},3}_{(4)} & 0 \end{pmatrix}, \tag{2.9}$$

where $\mathcal{Q}^{(4)}_{\alpha,1}$, $\mathcal{Q}^{(4)}_{\alpha,2}$, $\mathcal{Q}^{(4)}_{\alpha,3}$, and $\mathcal{Q}^{(4)}_{\alpha,4}$ are the usual 4-dimensional supercharges that belong to the $(\mathbf{1/2}, \mathbf{0})$ or $\mathbf{2}$ spinor representation of the $\mathrm{SO}(3,1)$, and their charge conjugations $\bar{\mathcal{Q}}^{\dot{\alpha},1}_{(4)}$, $\bar{\mathcal{Q}}^{\dot{\alpha},2}_{(4)}$, $\bar{\mathcal{Q}}^{\dot{\alpha},3}_{(4)}$, and $\bar{\mathcal{Q}}^{\dot{\alpha},4}_{(4)}$ belong to the $(\mathbf{0}, \mathbf{1/2})$ or $\mathbf{2}'$ spinor representation of the $\mathrm{SO}(3,1)$. So, we have the 4-dimensional $\mathcal{N} = 4$ supersymmetry. After the compactification, the $\mathrm{SO}(2)_{56}$ rotation on the x_5-x_6 plane, which is a subgroup of the $\mathrm{SO}(5,1)$, becomes the R symmetry in the 4-dimensional supersymmetric gauge theory. Therefore, the R symmetry in the 4-dimensional supersymmetric gauge theory is $\mathrm{SO}(2)_{56} \times \mathrm{SU}(2)_{\mathbf{4}_+} \times \mathrm{SU}(2)_{\mathbf{4}_-}$, which is the subgroup of $\mathrm{SU}(4)_R$, the maximal R symmetry in the 4-dimensional $\mathcal{N} = 4$ supersymmetric gauge theory. $(\mathcal{Q}^{(4)}_{\alpha,1}, \mathcal{Q}^{(4)}_{\alpha,2})^T$ form a fundamental representation of $\mathrm{SU}(2)_{\mathbf{4}_+}$, and $(\mathcal{Q}^{(4)}_{\alpha,3}, \mathcal{Q}^{(4)}_{\alpha,4})^T$ form a fundamental representation of $\mathrm{SU}(2)_{\mathbf{4}_-}$ where T is the transpose. Moreover, since the supercharges transform under the $\mathrm{SO}(5,1)$ as spinor representations, they are charged under the $\mathrm{SO}(2)_{56}$. From the 6-dimensional chiralities, we obtain that $\mathcal{Q}^{(4)}_{\alpha,1}$ and $\mathcal{Q}^{(4)}_{\alpha,2}$ carry the $\mathrm{SO}(2)_{56}$ charges $+1/2$, and $\mathcal{Q}^{(4)}_{\alpha,3}$ and $\mathcal{Q}^{(4)}_{\alpha,4}$ carry charges $-1/2$.

In the 6-dimensional $\mathcal{N} = (1,1)$ supersymmetric gauge theory, the vector multiplet is a unique supermultiplet of the $(1,1)$ supersymmetric theory besides the gravity multiplet. It consists of a 6-dimensional vector field A_μ ($\mu = 0,1,2,3$), A_5, A_6, two pseudo-Majorana-Weyl spinors with opposite chiralities $\lambda_{\mathbf{4}_-,A}$ and $\lambda_{\mathbf{4}_+,B}$ where $A, B = 1, 2$, and two complex scalar fields σ^2 and σ^3. All of them are in the adjoint representation under the bulk gauge group. The vector fields A_μ, A_5, and A_6 are invariant under the R-symmetry $\mathrm{SU}(2)_{\mathbf{4}_+} \times \mathrm{SU}(2)_{\mathbf{4}_-}$. The spinors $(\lambda_{\mathbf{4}_-,1}, \lambda_{\mathbf{4}_-,2})^T$ transform as a fundamental representation under the $\mathrm{SU}(2)_{\mathbf{4}_+}$, and the spinors $(\lambda_{\mathbf{4}_+,1}, \lambda_{\mathbf{4}_+,2})^T$ transform as a fundamental representation under the $\mathrm{SU}(2)_{\mathbf{4}_-}$. Four real components of scalar fields σ^2 and σ^3 belong to the $(\bar{\mathbf{2}}, \bar{\mathbf{2}})$ representation under R-symmetry $\mathrm{SU}(2)_{\mathbf{4}_+} \times \mathrm{SU}(2)_{\mathbf{4}_-}$, which transforms as

$$\Delta_{AB} \longrightarrow [\mathrm{SU}(2)_{\mathbf{4}_+}]_A^{\ A'} \, \Delta_{A'B'} \, [\mathrm{SU}(2)_{\mathbf{4}_-}]_B^{\ B'} , \tag{2.10}$$

where

$$\Delta_{AB} = \epsilon_{A'A} \begin{pmatrix} \sigma^2 & \sigma^3 \\ \sigma^{3*} & -\sigma^{2*} \end{pmatrix}^{A'B'} \epsilon_{B'B} \, . \tag{2.11}$$

So, they form a vector of the $\mathrm{SO}(4)_{\mathrm{R}} \simeq \mathrm{SU}(2)_{\mathbf{4}_+} \times \mathrm{SU}(2)_{\mathbf{4}_-}$.

If the 6-dimensional $\mathcal{N} = (1,1)$ supersymmetric gauge theory is compactified on a torus $\mathbf{T}^2$, we have 4-dimensional $\mathcal{N} = 4$ supersymmetry due to the flat metric of torus. The 4-dimensional $\mathcal{N} = 4$ vector multiplet contains a 4-dimensional vector field A_μ, four fermion partners χ^a_α $(a = 1, \ldots, 4)$, and three complex scalar fields $\sigma^1, \sigma^2, \sigma^3$. The four Weyl fermions χ^a_α are obtained from the 6-dimensional fermion fields $\lambda_{\mathbf{4}_-,\mathbf{A}}$ and $\lambda_{\mathbf{4}_+,\mathbf{B}}$ as the following decomposition

$$\lambda_{\mathbf{4}_-,1} \to \begin{pmatrix} 0 & -\chi^2_\alpha \\ \bar{\chi}^{\dot{\alpha}}_1 & 0 \end{pmatrix}, \qquad \lambda_{\mathbf{4}_-,2} \to \begin{pmatrix} 0 & \chi^1_\alpha \\ \bar{\chi}^{\dot{\alpha}}_2 & 0 \end{pmatrix}, \tag{2.12}$$

$$\lambda_{\mathbf{4}_+,1} \to \begin{pmatrix} \chi^4_\alpha & 0 \\ 0 & -\bar{\chi}^{\dot{\alpha}}_3 \end{pmatrix}, \qquad \lambda_{\mathbf{4}_+,2} \to \begin{pmatrix} -\chi^3_\alpha & 0 \\ 0 & -\bar{\chi}^{\dot{\alpha}}_4 \end{pmatrix}, \tag{2.13}$$

where the $(\chi^1_\alpha, \chi^2_\alpha)^T$ and $(\chi^3_\alpha, \chi^4_\alpha)^T$ transform as the anti-fundamental representations under the $\mathrm{SU}(2)_{\mathbf{4}_+}$ and $\mathrm{SU}(2)_{\mathbf{4}_-}$, respectively. And the complex field σ_1 is obtained from the gauge fields A_5 and A_6

$$\sigma^1(x^\mu, x^5, x^6) \equiv \frac{1}{\sqrt{2}} \left(A_6(x^\mu, x^5, x^6) + i A_5(x^\mu, x^5, x^6) \right), \tag{2.14}$$

which has charge -1 under the $\mathrm{SO}(2)_{56}$ symmetry.

In short, in terms of the 4-dimensional $\mathcal{N} = 1$ supersymmetry language, the 4-dimensional $\mathcal{N} = 4$ vector multiplet consists of one vector multiplet $V = (A_\mu, \chi^1)$ and three chiral multiplets $\Sigma_1 = (\sigma^1, \chi^2)$, $\Sigma_2 = (\sigma^2, \chi^3)$ and $\Sigma_3 = (\sigma^3, \chi^4)$ in the adjoint representation.

2.3 6-dimensional $\mathcal{N} = (1,1)$ supersymmetry breaking on T^2 orbifolds

Because we can not keep the 4-dimensional $\mathcal{N} = 4$ supersymmetry at low energy from 6-dimensional $\mathcal{N} = (1,1)$ supersymmetry, we have to break it down to the 4-dimensional $\mathcal{N} = 1$ supersymmetry by orbifold projections from the phenomenological point of view.

Before we discuss the 6-dimensional $\mathcal{N} = (1,1)$ supersymmetry reaking on T^2 orbifolds, we briefly review the symmetry breaking on T^2 orbifolds. Suppose Γ is a gauge (local) or global symmetry of the lagrangian and consider Z_n $(n = 3, 4, 6)$ discrete symmetry on T^2, we have

$$z \sim \omega z\,, \qquad \mathcal{L}(x^\mu, \omega z, \omega^{-1} \bar{z}) = \mathcal{L}(x^\mu, z, \bar{z})\,, \tag{2.15}$$

where $\omega = e^{\mathrm{i}2\pi/n}$. So, for a generic bulk multiplet Φ which fills a representation of the bulk symmetry group Γ, we have

$$\Phi(x^\mu, \omega z, \omega^{-1} \bar{z}) = \eta_\Phi (R^\Gamma_\omega)^{l_\Phi} \Phi(x^\mu, z, \bar{z}) ((R^\Gamma_\omega)^{-1})^{m_\Phi}\,, \tag{2.16}$$

where η_Φ, which is a possible phase for the bulk multiplet Φ from lagrangian, is an element of Z_n. And l_Φ and m_Φ are respectively the numbers of the fundamental index and anti-fundamental index for the bulk multiplet Φ under the bulk symmetry group Γ. For example, if Γ is a $\mathrm{SU}(N)$ group, for fundamental representation, $l_\Phi = 1$, $m_\Phi = 0$, and for adjoint

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representation, $l_\Phi = 1$, $m_\Phi = 1$. Moreover, R^Γ_ω is the representation of the Z_n symmetry element $z \to \omega z$, which is introduced to break the bulk symmetry group Γ. Also, R^Γ_ω is an element in Γ, and satisfies $(R^\Gamma_\omega)^n = 1$ due to $\omega^n = 1$. Because we disscuss the supersymmetry breaking and R symmetry breaking in this subsection, we choose $\eta_\Phi = 1$ in the following discussions.

Let us discuss the supersymmetry and R symmetry breakings. First, we consider the simple case where $R_\omega^{\mathrm{SU}(2)_{\mathbf{4}_+}}$ and $R_\omega^{\mathrm{SU}(2)_{\mathbf{4}_-}}$ are trivial

$$R_\omega^{\mathrm{SU}(2)_{\mathbf{4}_+}} = \mathrm{diag}[+1, +1]\,, \qquad R_\omega^{\mathrm{SU}(2)_{\mathbf{4}_-}} = \mathrm{diag}[+1, +1]\,. \tag{2.17}$$

Under Z_n symmetry

$$\gamma_n : \quad z \to \omega z\,, \tag{2.18}$$

i.e., $\mathrm{SO}(2)_{56}$ rotation with angle $2\pi/n$, we obtain that

$$A_\mu(x^\mu, \omega z, \omega^{-1}\bar{z}) = A_\mu(x^\mu, z, \bar{z})\,, \quad \sigma^1(x^\mu, \omega z, \omega^{-1}\bar{z}) = \omega^{-1}\sigma^1(x^\mu, z, \bar{z})\,, \tag{2.19}$$
$$\sigma^2(x^\mu, \omega z, \omega^{-1}\bar{z}) = \sigma^2(x^\mu, z, \bar{z})\,, \quad \sigma^3(x^\mu, \omega z, \omega^{-1}\bar{z}) = \sigma^3(x^\mu, z, \bar{z})\,, \tag{2.20}$$
$$\mathcal{Q}^{(4)}_{\alpha,1}(x^\mu, \omega z, \omega^{-1}\bar{z}) = \omega^{1/2}\mathcal{Q}^{(4)}_{\alpha,1}(x^\mu, z, \bar{z})\,, \tag{2.21}$$
$$\mathcal{Q}^{(4)}_{\alpha,2}(x^\mu, \omega z, \omega^{-1}\bar{z}) = \omega^{1/2}\mathcal{Q}^{(4)}_{\alpha,2}(x^\mu, z, \bar{z})\,, \tag{2.22}$$
$$\mathcal{Q}^{(4)}_{\alpha,3}(x^\mu, \omega z, \omega^{-1}\bar{z}) = \omega^{-1/2}\mathcal{Q}^{(4)}_{\alpha,3}(x^\mu, z, \bar{z})\,, \tag{2.23}$$
$$\mathcal{Q}^{(4)}_{\alpha,4}(x^\mu, \omega z, \omega^{-1}\bar{z}) = \omega^{-1/2}\mathcal{Q}^{(4)}_{\alpha,4}(x^\mu, z, \bar{z})\,, \tag{2.24}$$
$$\chi^1(x^\mu, \omega z, \omega^{-1}\bar{z}) = \omega^{-1/2}\chi^1(x^\mu, z, \bar{z})\,, \quad \chi^2(x^\mu, \omega z, \omega^{-1}\bar{z}) = \omega^{-1/2}\chi^2(x^\mu, z, \bar{z})\,, \tag{2.25}$$
$$\chi^3(x^\mu, \omega z, \omega^{-1}\bar{z}) = \omega^{1/2}\chi^3(x^\mu, z, \bar{z})\,, \quad \chi^4(x^\mu, \omega z, \omega^{-1}\bar{z}) = \omega^{1/2}\chi^4(x^\mu, z, \bar{z})\,. \tag{2.26}$$

From eqs. (2.21)–(2.26), we obtain that the 4-dimensional $\mathcal{N} = 4$ supersymmetry is completely broken.

In order to preserve the 4-dimensional $\mathcal{N} = 1$ supersymmetry and make sure that all the bulk fields have KK modes, we choose

$$R_\omega^{\mathrm{SU}(2)_{\mathbf{4}_+}} = \mathrm{diag}[\omega^{-1/2}, \omega^{+1/2}]\,, \qquad R_\omega^{\mathrm{SU}(2)_{\mathbf{4}_-}} = \mathrm{diag}[\omega^{3/2+k}, \omega^{-3/2-k}]\,, \tag{2.27}$$

where k is an integer, and $0 \le k < n$. Then, we have

$$V(x^\mu, \omega z, \omega^{-1}\bar{z}) = V(x^\mu, z, \bar{z})\,, \qquad \Sigma_1(x^\mu, \omega z, \omega^{-1}\bar{z}) = \omega^{-1}\Sigma_1(x^\mu, z, \bar{z})\,, \tag{2.28}$$
$$\Sigma_2(x^\mu, \omega z, \omega^{-1}\bar{z}) = \omega^{-1-k}\Sigma_2(x^\mu, z, \bar{z})\,, \qquad \Sigma_3(x^\mu, \omega z, \omega^{-1}\bar{z}) = \omega^{2+k}\Sigma_3(x^\mu, z, \bar{z})\,. \tag{2.29}$$

From eqs. (2.28) and (2.29), we obtain that for the zero modes, the 4-dimensional $\mathcal{N} = 4$ supersymmetry is broken down to the 4-dimensional $\mathcal{N} = 1$ supersymmetry if and only if $1 + k \neq 0$ mod n and $2 + k \neq 0$ mod n, because in this case only the gauge multiplet V has zero modes.

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From eq. (2.27), we obtain that for $k = 0$ and Z_3 discrete symmetry (T^2/Z_3 orbifold), the R symmetry is $\mathrm{SO}(2)_{56} \times \mathrm{U}(1)_{\mathbf{4}_+} \times \mathrm{SU}(2)_{\mathbf{4}_-}$ because $R_\omega^{\mathrm{SU}(2)_{\mathbf{4}_+}}$ breaks the $\mathrm{SU}(2)_{\mathbf{4}_+}$ R symmetry down to $\mathrm{U}(1)_{\mathbf{4}_+}$, and $R_\omega^{\mathrm{SU}(2)_{\mathbf{4}_-}}$ does not break the $\mathrm{SU}(2)_{\mathbf{4}_-}$ R symmetry. And for the rest cases, the R symmetry is $\mathrm{SO}(2)_{56} \times \mathrm{U}(1)_{\mathbf{4}_+} \times \mathrm{U}(1)_{\mathbf{4}_-}$ because $R_\omega^{\mathrm{SU}(2)_{\mathbf{4}_+}}$ and $R_\omega^{\mathrm{SU}(2)_{\mathbf{4}_-}}$ breaks the $\mathrm{SU}(2)_{\mathbf{4}_+} \times \mathrm{SU}(2)_{\mathbf{4}_-}$ R symmetry down to $\mathrm{U}(1)_{\mathbf{4}_+} \times \mathrm{U}(1)_{\mathbf{4}_-}$.

In addition, to preserve the 4-dimensional $\mathcal{N} = 1$ supersymmetry, we find that for the T^2/Z_3 orbifold, the only possibility is $k = 0$. And in this case, the R symmetry is $\mathrm{SO}(2)_{56} \times \mathrm{U}(1)_{\mathbf{4}_+} \times \mathrm{SU}(2)_{\mathbf{4}_-}$. Moreover, for the T^2/Z_4 orbifold, $k = 0$ is equivalent to $k = 1$, i.e., there is also only one possibility. For the T^2/Z_6 orbifold, the inequivalent cases are $k = 0$ and $k = 1$. The R symmetry for T^2/Z_4 and T^2/Z_6 orbifolds is $\mathrm{SO}(2)_{56} \times \mathrm{U}(1)_{\mathbf{4}_+} \times \mathrm{U}(1)_{\mathbf{4}_-}$.

The maximal R symmetry for the 4-dimensional $\mathcal{N} = 4$ supersymmetry is $\mathrm{SU}(4)_R$. Embedding the $\mathrm{SO}(2)_{56} \times \mathrm{SU}(2)_{\mathbf{4}_+} \times \mathrm{SU}(2)_{\mathbf{4}_-}$ R symmetry into the $\mathrm{SU}(4)_R$ gives us an elegant way to describe the supersymmetry and R symmetry breakings, so, we study it in detail.

The generator of $\mathrm{SO}(2)_{56}$ in $\mathrm{SU}(4)_R$ is

$$T_{\mathrm{SO}(2)_{56}} = \mathrm{diag}\left[+\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right], \tag{2.30}$$

and the generators of $\mathrm{SU}(2)_{\mathbf{4}_+} \times \mathrm{SU}(2)_{\mathbf{4}_-}$ are embedded into $\mathrm{SU}(4)_R$ as

$$\begin{pmatrix} T_{\mathrm{SU}(2)_{\mathbf{4}_+}} & 0 \\ 0 & T_{\mathrm{SU}(2)_{\mathbf{4}_-}} \end{pmatrix} \subset \left(T_{\mathrm{SU}(4)_R}\right). \tag{2.31}$$

We define

$$\left(\mathcal{Q}_\alpha^{(4)}\right)^T \equiv \left(\mathcal{Q}_{\alpha,1}^{(4)}, \mathcal{Q}_{\alpha,2}^{(4)}, \mathcal{Q}_{\alpha,3}^{(4)}, \mathcal{Q}_{\alpha,4}^{(4)}\right), \tag{2.32}$$

$$(\chi_\alpha)^T \equiv \left(\chi_\alpha^1, \chi_\alpha^2, \chi_\alpha^3, \chi_\alpha^4\right), \tag{2.33}$$

$$\sigma^{ab} \equiv \begin{pmatrix} 0 & \sigma^1 & \sigma^2 & \sigma^3 \\ -\sigma^1 & 0 & \sigma^{3*} & -\sigma^{2*} \\ -\sigma^2 & -\sigma^{3*} & 0 & \sigma^{1*} \\ -\sigma^3 & \sigma^{2*} & -\sigma^{1*} & 0 \end{pmatrix}. \tag{2.34}$$

And then, $\mathcal{Q}_\alpha^{(4)}$ transforms as the fundamental representation $\mathbf{4}$ under $\mathrm{SU}(4)_R$, χ_α transforms as the anti-fundamental representation $\bar{\mathbf{4}}$ under $\mathrm{SU}(4)_R$, and σ^{ab} transforms as the anti-symmetric tensor $\bar{\mathbf{4}} \wedge \bar{\mathbf{4}}$ of two anti-fundamental representations under $\mathrm{SU}(4)_R$.

Similar to the weakly-coupled heterotic string compactification, in order to preserve the 4-dimensional $\mathcal{N} = 1$ supersymmetry, $R_\omega^{\mathrm{SU}(4)_R}$ should take the following form

$$R_\omega^{\mathrm{SU}(4)_R} = \begin{pmatrix} 1 & 0 \\ 0 & R_\omega^{\mathrm{SU}(3)_R} \end{pmatrix}. \tag{2.35}$$

The supercharges $(\mathcal{Q}_{\alpha,2}^{(4)}, \mathcal{Q}_{\alpha,3}^{(4)}, \mathcal{Q}_{\alpha,4}^{(4)})^T$ form a fundamental representation under the $\mathrm{SU}(3)_R$, $(\chi_\alpha^2, \chi_\alpha^3, \chi_\alpha^4)^T$ and $(\sigma^1, \sigma^2, \sigma^3)^T$ form the anti-fundamental representation under the $\mathrm{SU}(3)_R$.

With eqs. (2.27), (2.30) and (2.31), we obtain

$$\begin{aligned} R_\omega^{\mathrm{SU}(4)_R} &= \mathrm{diag}[\omega^{+1/2}, \omega^{+1/2}, \omega^{-1/2}, \omega^{-1/2}] \times \mathrm{diag}[\omega^{-1/2}, \omega^{+1/2}, 1, 1] \times \\ &\quad \times \mathrm{diag}[1, 1, \omega^{3/2+k}, \omega^{-3/2-k}] \,. \end{aligned} \tag{2.36}$$

Thus, we have

$$R_\omega^{\mathrm{SU}(4)_R} = \mathrm{diag}[+1, \omega, \omega^{1+k}, \omega^{-2-k}] \,. \tag{2.37}$$

Similar to above discussions, for the zero modes, the 4-dimensional $\mathcal{N}=4$ supersymmetry is broken down to the 4-dimensional $\mathcal{N}=1$ supersymmetry if and only if $1+k \neq 0$ mod n and $2+k \neq 0$ mod n. And the transformations of vector multiplet under Z_n are given by eqs. (2.28) and (2.29).

Suppose G is a Lie group and H is a subgoup of G, for our convention, we denote the commutant of H in G as G/H, i.e.,

$$G/H \equiv \{g \in G | gh = hg, \text{ for any } h \in H\} \,. \tag{2.38}$$

With this convention, the unbroken R symmetry is given by

$$\left(\frac{\mathrm{SU}(4)_R}{R_\omega^{\mathrm{SU}(4)_R}} \right) \cap (\mathrm{SO}(2)_{56} \times \mathrm{SU}(2)_{\mathbf{4}_+} \times \mathrm{SU}(2)_{\mathbf{4}_-}) \,. \tag{2.39}$$

Similar to above results, to preserve the 4-dimensional $\mathcal{N}=1$ supersymmetry, we obtain that for T^2/Z_3 orbifold, the only possibility is $k=0$ and the R symmetry is $\mathrm{SO}(2)_{56} \times \mathrm{U}(1)_{\mathbf{4}_+} \times \mathrm{SU}(2)_{\mathbf{4}_-}$. And for T^2/Z_4 orbifold, the only possibility is $k=0$, too. For the T^2/Z_6 orbifold, there are two possibilities: $k=0$ and $k=1$. The R symmetry for T^2/Z_4 and T^2/Z_6 orbifolds is $\mathrm{SO}(2)_{56} \times \mathrm{U}(1)_{\mathbf{4}_+} \times \mathrm{U}(1)_{\mathbf{4}_-}$.

To be concrete, we define the generators for $\mathrm{U}(1)_{\mathbf{4}_+}$ and $\mathrm{U}(1)_{\mathbf{4}_-}$ in the $\mathrm{SU}(4)_R$

$$T_{\mathrm{U}(1)_{\mathbf{4}_+}} \equiv \mathrm{diag}\left[+\frac{1}{2}, -\frac{1}{2}, 0, 0\right] , \tag{2.40}$$

$$T_{\mathrm{U}(1)_{\mathbf{4}_-}} \equiv \mathrm{diag}\left[0, 0, +\frac{1}{2}, -\frac{1}{2}\right] . \tag{2.41}$$

To be complete, for the bulk gauge group G, we write down the bulk action in the Wess-Zumino gauge and 4-dimensional $\mathcal{N}=1$ supersymmetry language [32, 33]

$$\begin{aligned} S = \int d^6x \Bigg\{ &\mathrm{Tr}\left[\int d^2\theta \left(\frac{1}{4kg^2} \mathcal{W}^\alpha \mathcal{W}_\alpha + \frac{1}{kg^2} \left(\Sigma_3 \partial \Sigma_2 - \frac{1}{\sqrt{2}} \Sigma_1 [\Sigma_2, \Sigma_3] \right) \right) + \mathrm{h.c.} \right] + \\ &+ \int d^4\theta \frac{1}{kg^2} \mathrm{Tr}\left[(\sqrt{2}\partial^\dagger + \Sigma_1^\dagger) e^{-V} (-\sqrt{2}\partial + \Sigma_1) e^V \right] + \\ &+ \int d^4\theta \frac{1}{kg^2} \mathrm{Tr}\left[\Sigma_2^\dagger e^{-V} \Sigma_2 e^V + \Sigma_3{}^\dagger e^{-V} \Sigma_3 e^V \right] \Bigg\} \,. \end{aligned} \tag{2.42}$$

From above action, we obtain the transformations of vector multiplet

$$V(x^\mu, \omega z, \omega^{-1}\bar{z}) = (R_\omega^G)^{l_V} V(x^\mu, z, \bar{z})((R_\omega^G)^{-1})^{m_V}, \tag{2.43}$$

$$\Sigma_1(x^\mu, \omega z, \omega^{-1}\bar{z}) = \omega^{-1}(R_\omega^G)^{l_{\Sigma_1}} \Sigma_1(x^\mu, z, \bar{z})((R_\omega^G)^{-1})^{m_{\Sigma_1}}, \tag{2.44}$$

$$\Sigma_2(x^\mu, \omega z, \omega^{-1}\bar{z}) = \omega^{-1-k}(R_\omega^G)^{l_{\Sigma_2}} \Sigma_2(x^\mu, z, \bar{z})((R_\omega^G)^{-1})^{m_{\Sigma_2}}, \tag{2.45}$$

$$\Sigma_3(x^\mu, \omega z, \omega^{-1}\bar{z}) = \omega^{2+k}(R_\omega^G)^{l_{\Sigma_3}} \Sigma_3(x^\mu, z, \bar{z})((R_\omega^G)^{-1})^{m_{\Sigma_3}}, \tag{2.46}$$

where we introduce the non-trivial R_ω^G to break the bulk gauge group G. When we consider the gauge symmetry breaking by discrete symmetry on the extra space manifold, we can decompose the gauge fields of G under its maximal subgroup, and then, discuss the gauge symmetry breaking, for example, the E_6 breaking and E_8 breaking in refs. [25] and [26], respectively. Some of the gauge fields of G are not in the adjoint representations under its maximal subgroup due to decomposition, so, we write down the explicit l_Φ and m_Φ for the general orbifold gauge symmetry breaking.

Moreover, under the $\mathrm{SO}(2)_{56}$ R symmetry, the θ (the Grassmann coordinate for the unbroken 4-dimensional $\mathcal{N} = 1$ supersymmetry), z and the bulk vector multiplet transform as

$$z \longrightarrow e^{\mathrm{i}\beta} z, \qquad \theta \longrightarrow e^{-\mathrm{i}\beta/2}\theta, \qquad V \longrightarrow V, \tag{2.47}$$

$$\Sigma_1 \longrightarrow e^{-\mathrm{i}\beta}\Sigma_1, \qquad \Sigma_2 \longrightarrow \Sigma_2, \qquad \Sigma_3 \longrightarrow \Sigma_3. \tag{2.48}$$

Under the $\mathrm{U}(1)_{\mathbf{4}_+}$ R symmetry, the Grassmann coordinate θ, z and the bulk vector multiplet transform as

$$z \longrightarrow z, \qquad \theta \longrightarrow e^{-\mathrm{i}\beta/2}\theta, \qquad V \longrightarrow V, \tag{2.49}$$

$$\Sigma_1 \longrightarrow \Sigma_1, \qquad \Sigma_2 \longrightarrow e^{-\mathrm{i}\beta/2}\Sigma_2, \qquad \Sigma_3 \longrightarrow e^{-\mathrm{i}\beta/2}\Sigma_3. \tag{2.50}$$

For the T^2/Z_3 orbifold, $(\Sigma_2, \Sigma_3)^T$ form an anti-fundamental representation under $\mathrm{SU}(2)_{\mathbf{4}_-}$. And for the T^2/Z_4 and T^2/Z_6 orbifolds, Σ_2 and Σ_3 have the $\mathrm{U}(1)_{\mathbf{4}_-}$ charges $-1/2$ and $+1/2$, respectively, while θ, z, V and Σ_1 are neutral.

3. $\mathcal{N} = (1,1)$ supersymmetric E_6 models on $M^4 \times T^2/Z_3$

In this section, we discuss the $\mathcal{N} = (1,1)$ supersymmetric E_6 models on the space-time $M^4 \times T^2/Z_3$ where the gauge symmetry E_6 is broken down to $\mathrm{SO}(10) \times \mathrm{U}(1)_X$ by orbifold projection. And there are three SO(10) spinor representation **16**s from the zero modes of bulk vector multiplet, which can be considered as three families of fermions. We will discuss two models in detail, and show that the fermion masses and mixings can be explained by the flavour symmetry $\mathrm{SU}(2) \times \mathrm{U}(1)$ whose origin is R symmetry. By the way, in this section, we consider the T^2/Z_3 orbifold, so, $\omega = e^{\mathrm{i}2\pi/3}$.

The gauge fields of E_6 are in the adjoint representation with dimension **78**. Under the gauge group $\mathrm{SO}(10) \times \mathrm{U}(1)_X$, the E_6 gauge fields decompose as [34]

$$\mathbf{78} = (\mathbf{45}, \mathbf{0}) \oplus (\mathbf{16}, \mathbf{-3}) \oplus (\overline{\mathbf{16}}, \mathbf{3}) \oplus (\mathbf{1}, \mathbf{0}). \tag{3.1}$$

The $R_\omega^{E_6}$ for E_6 gauge group is the product of the $R_\omega^{\mathrm{SO}(10)}$ for SO(10) and $R_\omega^{\mathrm{U}(1)_X}$ for $\mathrm{U}(1)_X$

$$R_\omega^{E_6} = R_\omega^{\mathrm{SO}(10)} \otimes R_\omega^{\mathrm{U}(1)_X} . \tag{3.2}$$

In order to break the E_6 gauge symmetry, we choose

$$R_\omega^{\mathrm{SO}(10)} = \mathrm{diag}[+1,+1,+1,+1,+1,+1,+1,+1,+1,+1] , \tag{3.3}$$

$$R_\omega^{\mathrm{U}(1)_X} = \exp\left(-\frac{\mathrm{i}2\pi Q}{9}\right) , \tag{3.4}$$

where Q is the $\mathrm{U}(1)_X$ charge of the field. And the sufficient condition for $R_\omega^{\mathrm{U}(1)_X}$ to satisfy is that $(R_\omega^{\mathrm{U}(1)_X})^3$ is equal to 1 for all the bulk fields which are charged under $\mathrm{U}(1)_X$.

Using eqs. (2.43)–(2.46) with $k = 0$ and eqs. (3.2)–(3.4), we obtain the transformations of vector multiplet

$$V_{(\mathbf{45},\mathbf{0})}(x^\mu, \omega z, \omega^{-1}\bar{z}) = V_{(\mathbf{45},\mathbf{0})}(x^\mu, z, \bar{z}) , \tag{3.5}$$

$$V_{(\mathbf{1},\mathbf{0})}(x^\mu, \omega z, \omega^{-1}\bar{z}) = V_{(\mathbf{1},\mathbf{0})}(x^\mu, z, \bar{z}) , \tag{3.6}$$

$$V_{(\mathbf{16},-\mathbf{3})}(x^\mu, \omega z, \omega^{-1}\bar{z}) = \omega V_{(\mathbf{16},-\mathbf{3})}(x^\mu, z, \bar{z}) , \tag{3.7}$$

$$V_{(\overline{\mathbf{16}},\mathbf{3})}(x^\mu, \omega z, \omega^{-1}\bar{z}) = \omega^2 V_{(\overline{\mathbf{16}},\mathbf{3})}(x^\mu, z, \bar{z}) , \tag{3.8}$$

$$(\Sigma_i)_{(\mathbf{45},\mathbf{0})}(x^\mu, \omega z, \omega^{-1}\bar{z}) = \omega^2 (\Sigma_i)_{(\mathbf{45},\mathbf{0})}(x^\mu, z, \bar{z}) , \tag{3.9}$$

$$(\Sigma_i)_{(\mathbf{1},\mathbf{0})}(x^\mu, \omega z, \omega^{-1}\bar{z}) = \omega^2 (\Sigma_i)_{(\mathbf{1},\mathbf{0})}(x^\mu, z, \bar{z}) , \tag{3.10}$$

$$(\Sigma_i)_{(\mathbf{16},-\mathbf{3})}(x^\mu, \omega z, \omega^{-1}\bar{z}) = (\Sigma_i)_{(\mathbf{16},-\mathbf{3})}(x^\mu, z, \bar{z}) , \tag{3.11}$$

$$(\Sigma_i)_{(\overline{\mathbf{16}},\mathbf{3})}(x^\mu, \omega z, \omega^{-1}\bar{z}) = \omega (\Sigma_i)_{(\overline{\mathbf{16}},\mathbf{3})}(x^\mu, z, \bar{z}) , \tag{3.12}$$

where $i = 1, 2, 3$, and $\omega = e^{\mathrm{i}2\pi/3}$.

From the general discussions on gauge symmetry breaking by discrete symmetry [26], we obtain that only the fields $V_{(\mathbf{45},\mathbf{0})}$, $V_{(\mathbf{1},\mathbf{0})}$, and $(\Sigma_i)_{(\mathbf{16},-\mathbf{3})}$ have zero modes. So, the E_6 gauge symmetry is broken down to $\mathrm{SO}(10) \times \mathrm{U}(1)_X$ by orbifold projection for the zero modes. And there are three SO(10) spinor representations **16** from the zero modes of bulk vector multiplet, which can be considered as three families of fermions. This may explain why there are three families of fermions in the nature. Moreover, the R symmetry is $\mathrm{SO}(2)_{56} \times \mathrm{U}(1)_{\mathbf{4}_+} \times \mathrm{SU}(2)_{\mathbf{4}_-}$, which can be the origin of flavour symmetry to generate the fermion masses and mixings.

3.1 Model I

In model I, the zero modes $(\Sigma_i)^{(0)}_{(\mathbf{16},-\mathbf{3})}$ of $(\Sigma_i)_{(\mathbf{16},-\mathbf{3})}$ where $i = 1, 2, 3$ are considered as three families of fermions, and the $\mathrm{SO}(2)_{56} \times \mathrm{U}(1)_{\mathbf{4}_+} \times \mathrm{SU}(2)_{\mathbf{4}_-}$ R symmetry can give us the flavour symmetry to generate the fermion masses and mixings.

To be convenient in discussions, we define

$$\mathbf{16}_1 \equiv -(\Sigma_3)^{(0)}_{(\mathbf{16},-\mathbf{3})} , \qquad \mathbf{16}_2 \equiv (\Sigma_2)^{(0)}_{(\mathbf{16},-\mathbf{3})} , \qquad \mathbf{16}_3 \equiv (\Sigma_1)^{(0)}_{(\mathbf{16},-\mathbf{3})} , \tag{3.13}$$

where $\mathbf{16}_i$ is the $i-th$ family of the Standard Model fermions. In this convention, the first two families $(\mathbf{16}_1, \mathbf{16}_2)^T$ form a fundamental representation under the R symmetry $\mathrm{SU}(2)_{\mathbf{4}_-}$.

In order to give the same $\mathrm{U}(1)_R$ charges for three families of fermions $\mathbf{16}_i$, we define the $\mathrm{U}(1)_R^I$ and $\mathrm{U}(1)_F^I$ global symmetries from the $\mathrm{SO}(2)_{56} \times \mathrm{U}(1)_{\mathbf{4}_+}$ R symmetry, whose generators in $\mathrm{SU}(4)_R$ are

$$T_{\mathrm{U}(1)_R^I} = \mathrm{diag}\left[+\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right], \tag{3.14}$$

$$T_{\mathrm{U}(1)_F^I} = \mathrm{diag}\left[0, -1, +\frac{1}{2}, +\frac{1}{2}\right]. \tag{3.15}$$

So, under $\mathrm{U}(1)_R^I$, the Grassmann coordinate θ has charge $-3/2$, and three families of fermions (superfields) $\mathbf{16}_i$ have the same charge -1. Then, the superpotential has $\mathrm{U}(1)_R^I$ charge -3. In addition, under $\mathrm{U}(1)_F^I$, the Grassmann coordinate θ has charge 0, $\mathbf{16}_3$ has charge $+1$, and $\mathbf{16}_1$ and $\mathbf{16}_2$ have charge $-1/2$.

In Model I, the R symmetry $\mathrm{U}(1)_F^I \times \mathrm{SU}(2)_{\mathbf{4}_-}$ can be considered as flavour symmetry. People have discussed the fermion masses and mixings extensively in the supersymmetric SO(10) models with U(2) or $\mathrm{SU}(2) \times \mathrm{U}(1)^n$ flavour symmetries [11, 12, 13, 14]. In this paper, we follow the discussions in ref. [12], where the flavour symmetry is $\mathrm{SU}(2) \times \mathrm{U}(1)$ and the fermion masses and mixings can be explained.

To simplify the convention, we denote $\mathbf{16}_1$ and $\mathbf{16}_2$ as $\mathbf{16}_{a'}$ where the low index a' is the $\mathrm{SU}(2)_{\mathbf{4}_-}$ index for fundamental representation. And the upper index a' is the $\mathrm{SU}(2)_{\mathbf{4}_-}$ index for anti-fundamental representation. We also assume that the particles, which are introduced to break the gauge symmetry and generate the fermion masses and mixings, are on the observable 3-brane at the fixed point $z = 0$. Similar to the particle content in ref. [12], we introduce a $\mathbf{10}$ dimensional Higgs field H, a $\mathbf{45}$ dimensional Higgs field $\mathbf{45}_H$ which has a VEV along the $B-L$ direction, and one pair of $\mathbf{16}$ and $\overline{\mathbf{16}}$ Higgs fields (ψ_H, $\overline{\psi}_H$) which are assumed to have the VEVs along the "right-handed" neutrino direction and break the SO(10) gauge symmetry down to SU(5). $\overline{\psi}_H$ can generate the right-handed neutrino Majorana masses. Moreover, we introduce three pairs of heavy $\mathbf{16}$s and $\overline{\mathbf{16}}$s of SO(10): (ψ, $\overline{\psi}$), ($\psi^{a'}$, $\overline{\psi}_{a'}$), ($\psi_{a'}$, $\overline{\psi}^{a'}$). These heavy $\mathbf{16}$s and $\overline{\mathbf{16}}$s do not have VEVs, and we can obtain the non-renormalizable Yukawa matrices after we integrate them out. We also add three SO(10) singlets $N_{a'}$ and N_3.

In order to break the flavour symmetry $\mathrm{U}(1)_F^I \times \mathrm{SU}(2)_{\mathbf{4}_-}$, we introduce an anti-fundamental representation $\phi^{a'}$ (doublet), a symmetric representation with two anti-fundamental indices $S^{a'b'}$ (triplet), and an anti-symmetric representation with two anti-fundamental indices $A^{a'b'}$ (singlet) where the representations are under $\mathrm{SU}(2)_{\mathbf{4}_-}$ symmetry. In addition, $\phi^{a'}$, $S^{a'b'}$, and $A^{a'b'}$ are SO(10) singlets and charged under $\mathrm{U}(1)_F^I$.

The quantum numbers for above particles in Model I under $\mathrm{U}(1)_X$ gauge symmetry and $\mathrm{U}(1)_R^I \times \mathrm{U}(1)_F^I$ R symmetry are given in table 1 where Q_X, Q_R and Q_F are free parameters.

I. Low energy R symmetry. As we know, under the traditional $\mathrm{U}(1)_R$ symmetry defined in the low energy supersymmetry phenomenology, the Grassmann coordinate θ and the superfields $\mathbf{16}_i$ have charge $+1$, while the Higgs fields (H and $\mathbf{45}_H$) have charge zero. So, the $\mathrm{U}(1)_R^I$ symmetry defined in eq. (3.14) is not the traditional $\mathrm{U}(1)_R$ symmetry,

Particles	$\mathrm{U}(1)_X \times \mathrm{U}(1)_R^I \times \mathrm{U}(1)_F^I$
$\mathbf{16}_{a'}$	$(-3;\ -1;\ -1/2\ )$
$\mathbf{16}_3$	$(-3;\ -1;\ +1\ )$
H	$(+6;\ -1;\ -2\ )$
$\mathbf{45}_H$	$(-Q_X;\ -Q_R;\ -Q_F+3)$
ψ_H	$(0;\ Q^R_{\psi_H};\ Q^F_{\psi_H})$
$\overline{\psi}_H$	$(Q_X/2+3;\ Q_R/2-1/2;\ Q_F/2-5/2\ )$
ψ	$(-Q_X-3;\ -Q_R-1;\ -Q_F+4)$
$\overline{\psi}$	$(Q_X+3;\ Q_R-2;\ Q_F-4)$
$\psi^{a'}$	$(-3;\ -1;\ 5/2)$
$\overline{\psi}_{a'}$	$(+3;\ -2;\ -5/2)$
$\psi_{a'}$	$(-Q_X-3;\ -Q_R-1;\ -Q_F+5/2)$
$\overline{\psi}^{a'}$	$(Q_X+3;\ Q_R-2;\ Q_F-5/2)$
$N_{a'}$	$(-Q_X/2;\ -Q_R/2-3/2;\ -Q_F/2)$
N_3	$(-Q_X/2;\ -Q_R/2-3/2;\ -Q_F/2+3/2)$
$\phi^{a'}$	$(Q_X;\ Q_R;\ Q_F-3/2)$
$S^{a'b'}$	$(Q_X;\ Q_R;\ Q_F)$
$A^{a'b'}$	$(0;\ 0;\ +3)$

Table 1: Model I. Under the $\mathrm{U}(1)_X$ gauge symmetry and $\mathrm{U}(1)_R^I\times\mathrm{U}(1)_F^I$ R symmetry, the quantum numbers for the Standard Model fermions $\mathbf{16}_{a'}$, $\mathbf{16}_3$; Higgs fields H, $\mathbf{45}_H$, ψ_H, $\overline{\psi}_H$; vector like particles ψ, $\overline{\psi}$, $\psi^{a'}$, $\overline{\psi}_{a'}$, $\psi_{a'}$, $\overline{\psi}^{a'}$; three SO(10) singlets $N_{a'}$ and N_3; and Higgs fields for flavour symmetry breaking $\phi^{a'}$, $S^{a'b'}$, and $A^{a'b'}$. In this table, Q_X, Q_R and Q_F are free parameters. And the $\mathrm{U}(1)_R^I\times\mathrm{U}(1)_F^I$ charges for ψ_H ($Q^R_{\psi_H}$ and $Q^F_{\psi_H}$) are arbitrary because we have no constraints from superpotential.

and it is broken by the VEVs of H, ψ_H and $\overline{\psi}_H$ (The VEVs of $\phi^{a'}$, $S^{a'b'}$, and $A^{a'b'}$ will not break the $\mathrm{U}(1)_R^I$ symmetry if $Q_R=0$.). Moreover, there is no unbroken Z_2 symmetry under which the ordinary particles are even while their superpartners are odd. Thus, there are no cold dark matter candidates (for example neutralino) from the superpartners, although the axion can still be the cold dark matter candidate. However, this $\mathrm{U}(1)_R^I$ symmetry can still forbid the dimension-5 proton decay operators $\mathbf{16}_i\mathbf{16}_j\mathbf{16}_k\mathbf{16}_l$. And we would like to point out that this $\mathrm{U}(1)_R^I$ symmetry may be the unbroken U(1) R symmetry in the compactification of the weakly coupled heterotic string theory on Calabi-Yau manifolds. So, it is interesting to study its low energy phenomenology. In the next subsection, we will discuss a model where the first two families come from the zero modes of bulk vector multiplet and the U(1) R symmetry is the traditional $\mathrm{U}(1)_R$ R symmetry at low energy supersymmetry phenomenology.

II. Anomaly cancellations. In our model, there is no $[\mathrm{SO}(10)]^3$ anomaly. But, there exist the mixed anomalies $[\mathrm{SO}(10)]^2\,\mathrm{U}(1)_X$, $[\mathrm{U}(1)_X]^3$, and $[\mathrm{Gravity}]^2\,\mathrm{U}(1)_X$, which are localized on three 3-branes at three fixed points. All three anomalies can be cancelled by a generalized Green-Schwarz mechanism [35], and then, $\mathrm{U}(1)_X$ becomes a global symmetry which should be preserved by the superpotential localized on the 3-branes at the fixed points. However, the Fayet-Iliopoulos D-terms may be generated radiatively on three 3-branes at three fixed points [36]. To avoid the supersymmetry breaking from these Fayet-Iliopoulos D-terms, we require the $\mathrm{U}(1)_X$ D-flatnesses on three 3-branes at three fixed points. And to achieve the $\mathrm{U}(1)_X$ D-flatnesses, we introduce one pair of the SO(10) singlets with $\mathrm{U}(1)_X$ charges ± 1 on each 3-brane, and give them non-zero VEVs. Therefore, if we assume that the anomalies are cancelled by a generalized Green-Schwarz mechanism, we do not need to consider the $\mathrm{U}(1)_X$ anomalies and its breaking, although we require that the superpotential localized on the 3-branes at the fixed points preserve the $\mathrm{U}(1)_X$ symmetry.

In addition, we can cancel the anomalies by introducing the vector-like particles on the 3-branes at the fixed points. The anomalies due to the bulk vector multiplets arise from the zero modes of $(\Sigma_i)_{(\mathbf{16},\mathbf{-3})}$, i.e., three families of fermions $\mathbf{16}_{a'}$ and $\mathbf{16}_3$. And these anomalies distribute equally on three 3-branes at three fixed points since these fixed points are equivalent from geometry. Therefore, the anomalies on the 3-brane at each fixed point due to the bulk vector multiplet are equivalent to the anomalies from a multiplet with quantum number $(\mathbf{16},\mathbf{-3})$ under the $\mathrm{SO}(10)\times\mathrm{U}(1)_X$ gauge symmetry. And we just need to cancel the anomalies on the 3-brane at each fixed point.

To simplify the discussions, we choose $Q_X = 0$. First, on the observable 3-brane at $z = 0$, the anomalies due to the bulk vector multiplet are cancelled by the Higgs field $\overline{\psi}_H$. Then, we only need to cancel the anomalies generated by the Higgs field H. So, we introduce one pair of 10-plets T_{10} and T'_{10} whose quantum numbers are $(\mathbf{10},\mathbf{-6})$ and $(\mathbf{10},\mathbf{0})$ respectively under the $\mathrm{SO}(10)\times\mathrm{U}(1)_X$ gauge symmetry. In order to give the masses to T_{10} and T'_{10}, and keep the $\mathrm{U}(1)_X$ D-flatness and anomaly free on the 3-brane at $z = 0$, we introduce the SO(10) singlets s_1 and s'_1 with $\mathrm{U}(1)_X$ charges $+6$ and -6, respectively. In order to forbid the mixings between the Higgs H and T_{10} or between H and T'_{10}, we assign the following $\mathrm{U}(1)^I_R\times\mathrm{U}(1)^I_F$ charges for these vector-like particles: the $\mathrm{U}(1)^I_R$ charges for T_{10}, T'_{10}, s_1 and s'_1 are 0, 0, -3 and 0, respectively, and the $\mathrm{U}(1)^I_F$ charges for T_{10}, T'_{10}, s_1 and s'_1 are -1, -1, 2 and -1, respectively. One can check that the anomalies $[\mathrm{SO}(10)]^2\,\mathrm{U}(1)_X$, $[\mathrm{U}(1)_X]^3$, and $[\mathrm{Gravity}]^2\,\mathrm{U}(1)_X$ are cancelled on the 3-brane at $z = 0$. And the masses of T_{10} and T'_{10} can be generated by the superpotential $s_1 T_{10} T'_{10}$.

Second, on the 3-brane at $z = \pi R e^{\mathrm{i}\pi/6}/\sqrt{3}$, we need to cancel the anomalies generated by a multiplet with quantum number $(\mathbf{16},\mathbf{-3})$ under the $\mathrm{SO}(10)\times\mathrm{U}(1)_X$ gauge symmetry. So, we introduce a pair of the Υ_2 and Υ'_2 whose quantum numbers are $(\mathbf{16},\mathbf{0})$ and $(\overline{\mathbf{16}},\mathbf{+3})$ respectively under the $\mathrm{SO}(10)\times\mathrm{U}(1)_X$ gauge symmetry. And in order to give the masses to Υ_2 and Υ'_2, and keep the $\mathrm{U}(1)_X$ D-flatness and anomaly free on the 3-brane at $z = \pi R e^{\mathrm{i}\pi/6}/\sqrt{3}$, we introduce the SO(10) singlets s_2 and s'_2 with $\mathrm{U}(1)_X$ charges -3 and $+3$, respectively. To forbid the mixings between the SM fermions $\mathbf{16}_i$ and Υ_2 and the mass terms between $\mathbf{16}_i$ and Υ'_2, we assign the following $\mathrm{U}(1)^I_R\times\mathrm{U}(1)^I_F$ charges for the vector-like particles: the $\mathrm{U}(1)^I_R$ charges for Υ_2, Υ'_2, s_2 and s'_2 are 0, 0, -3 and 0, respectively, and the $\mathrm{U}(1)^I_F$ charges for Υ_2, Υ'_2, s_2 and s'_2 are $-1/2$, $-1/2$, 1 and -1, respectively. In short, the anomalies $[\mathrm{SO}(10)]^2\,\mathrm{U}(1)_X$, $[\mathrm{U}(1)_X]^3$, and $[\mathrm{Gravity}]^2\,\mathrm{U}(1)_X$ are cancelled on the 3-brane at $z = \pi R e^{\mathrm{i}\pi/6}/\sqrt{3}$. And the masses of Υ_2 and Υ'_2 can be generated by the superpotential $s_2\Upsilon_2\Upsilon'_2$.

Third, the anomaly cancellations on the 3-brane at $z = 2\pi R e^{\mathrm{i}\pi/6}/\sqrt{3}$ is similar to those on the 3-brane at $z = \pi R e^{\mathrm{i}\pi/6}/\sqrt{3}$. We introduce the particles Υ_3, Υ'_3, s_3 and s'_3, whose quantum numbers are the same as those for Υ_2, Υ'_2, s_2 and s'_2, respectively.

Furthermore, similar to the 5-dimensional orbifold theories [37] and string theory [38], it seems to us that there exist the anomaly inflows among the 3-branes at the fixed points. If this is true, like the 5-dimensional orbifold theories, the 4-dimensional anomaly cancellations are sufficient to ensure the consistency of higher dimensional orbifold theories [37]. In other words, we only need to consider the total anomalies due to the bulk zero modes and the particles localized on the 3-branes at the fixed points. For example, we can put Υ_2, Υ'_2, s_2, s'_2, Υ_3 and Υ'_3 on the 3-brane at $z = \pi R e^{\mathrm{i}\pi/6}/\sqrt{3}$. The anomalies are cancelled due to anomaly inflows, and Υ_3 and Υ'_3 can obtain the masses via the superpotential $s_2 \Upsilon_3 \Upsilon'_3$. The explicit proof of the anomaly inflows among the 3-branes at the fixed points on T^2 orbifolds is an interesting open question.

III. Fermion masses and mixings. The general VEVs for $\phi^{a'}$, $S^{a'b'}$, and $A^{a'b'}$, which break the $\mathrm{U}(1)^I_F \times \mathrm{SU}(2)_{\mathbf{4}_-}$ flavour symmetry, are

$$\langle \phi^2 \rangle = r \, \langle S^{22} \rangle \,, \qquad \langle A^{12} \rangle \neq 0 \,, \tag{3.16}$$

$$\langle S^{22} \rangle \neq 0 \,, \qquad \langle S^{11} \rangle = \kappa_1 \, \langle S^{22} \rangle \,, \qquad \langle S^{12} \rangle = \kappa_2 \, \langle S^{22} \rangle \,, \tag{3.17}$$

where the constants r, κ_1 and κ_2 are arbitrary.

The superpotential is

$$\begin{aligned} W = {} & 16_3 \, H \, 16_3 + 16_{a'} \, H \, \psi^{a'} + \overline{\psi} \, (M' \, \psi + 45_H \, 16_3) + \\ & + \overline{\psi}_{a'} \, (M \, \psi^{a'} + \phi^{a'} \, \psi + S^{a' b'} \, \psi_{b'} + A^{a' b'} \, 16_{b'}) + \overline{\psi}^{a'} \, (M'' \, \psi_{a'} + 45_H \, 16_{a'}) + \\ & + \overline{\psi}_H \, (N_{a'} \, \psi^{a'} + N_3 \, 16_3) + \frac{1}{2} \, N_{a'} \, N_{b'} \, S^{a' b'} + N_{a'} \, N_3 \, \phi^{a'} \,, \end{aligned} \tag{3.18}$$

where

$$M = M_0 \, (1 + \alpha_{X'} \, T_{X'} + \beta_Y \, T_Y) \,, \tag{3.19}$$

$$\alpha_{X'} \propto \frac{\langle \mathbf{45}_{X'} \rangle}{M_0} \,, \qquad \beta_Y \propto \frac{\langle \mathbf{45}_Y \rangle}{M_0} \,, \tag{3.20}$$

where $T_{X'}$ and T_Y are the generators for $\mathrm{U}(1)_{X'}$ and $\mathrm{U}(1)_Y$, respectively where the $\mathrm{U}(1)_{X'}$ is the U(1) in SO(10) which commutes with SU(5), and the $\mathrm{U}(1)_Y$ is the standard weak hypercharge interaction. The Higgs fields $\mathbf{45}_{X'}$ and $\mathbf{45}_Y$ have VEVs along the $\mathrm{U}(1)_{X'}$ and $\mathrm{U}(1)_Y$ directions, respectively. And $\alpha_{X'}$ and β_Y are arbitrary constants which are used to fit the data. Futhermore, each term in superpotential has an order 1 ($\mathcal{O}(1)$) coupling constant which is omitted for notational simplicity. And the effective mass parameters M_0, M', and M'' can be generated by the VEVs of the SO(10) singlet chiral superfields.

The mass scales are assumed to satisfy $M_0 \sim M' \sim M'' \gg \langle \phi^2 \rangle \sim \langle S^{22} \rangle \gg \langle A^{12} \rangle$ where M_0 may be at the order of GUT scale. And the small parameters in Yukawa matrices, which are used to explain the fermion masses and mixings, are $\epsilon \sim \langle S^{22} \rangle \langle 45 \rangle / M_0^2$ and $\epsilon' \sim \langle A^{12} \rangle / M_0$. In this model, the second generation masses are of order ϵ, while the first generation masses are of order ϵ'^2/ϵ.

In the effective theory below M_0, after we integrate out the Froggatt-Nielsen states $\{\psi,\, \bar\psi,\, \psi^{a'},\, \bar\psi_{a'},\, \psi_{a'},\, \bar\psi^{a'}\}$, we obtain the effective Yukawa matrices for up-type quarks, down-type quarks, and charged leptons, and the Dirac neutrino Yukawa matrix

$$Y_u = \begin{pmatrix} \kappa_1\,\epsilon\,\rho & (\epsilon' + \kappa_2\,\epsilon)\rho & 0 \\ -(\epsilon' - \kappa_2\,\epsilon)\rho & \epsilon\rho & \epsilon r\,T_{\bar u} \\ 0 & \epsilon r\,T_Q & 1 \end{pmatrix} y\,, \tag{3.21}$$

$$Y_d = \begin{pmatrix} \kappa_1\,\epsilon & \epsilon' + \kappa_2\,\epsilon & 0 \\ -(\epsilon' - \kappa_2\,\epsilon) & \epsilon & \epsilon r\delta\,T_{\bar d} \\ 0 & \epsilon r\,T_Q & 1 \end{pmatrix} y\,, \tag{3.22}$$

$$Y_e = \begin{pmatrix} 3\,\kappa_1\,\epsilon & -(\epsilon' - 3\,\kappa_2\,\epsilon) & 0 \\ \epsilon' + 3\,\kappa_2\,\epsilon & 3\epsilon & \epsilon r\,T_{\bar e} \\ 0 & \epsilon r\delta\,T_L & 1 \end{pmatrix} y\,, \tag{3.23}$$

$$Y_\nu = \begin{pmatrix} 3\,\kappa_1\,\epsilon\,\delta' & -(\epsilon' - 3\,\kappa_2\,\epsilon)\,\delta' & 0 \\ (\epsilon' + 3\,\kappa_2\,\epsilon)\,\delta' & 3\epsilon\delta' & \frac{1}{2}\epsilon r\delta'\,T_{\bar\nu} \\ 0 & \epsilon r\delta\,T_L & 1 \end{pmatrix} y\,, \tag{3.24}$$

where

$$\rho = -\frac{5\beta_Y(1-3\alpha_{X'})}{16\alpha_{X'}(1+\alpha_{X'})}\,, \qquad \delta = \frac{1+\alpha_{X'}}{1-3\alpha_{X'}}\,, \qquad \delta' = \frac{2\delta}{2\delta-1}\,, \tag{3.25}$$

$$T_f = (\text{Baryon Number} - \text{Lepton Number})\,, \qquad \text{where } f = \{Q, \bar u, \bar d, L, \bar e, \bar\nu\}\,. \tag{3.26}$$

In the basis $\{\nu_1, \nu_2, \nu_3, \bar\nu_1, \bar\nu_2, \bar\nu_3, N_1, N_2, N_3\}$, the generalized neutrino mass matrix is

$$M_{\nu,\bar\nu,N} = \begin{pmatrix} 0 & M_D^\nu & 0 \\ (M_D^\nu)^T & 0 & M_{\bar\nu,N} \\ 0 & (M_{\bar\nu,N})^T & M_N \end{pmatrix}, \tag{3.27}$$

where

$$(M_D^\nu)_{ij} = (Y_\nu)_{ij}\,\frac{v}{\sqrt{2}}\,\sin\beta\,, \tag{3.28}$$

$$M_{\bar\nu,N} = \begin{pmatrix} 3\,\kappa_1\,\epsilon\,\tilde V_{\overline{\psi}_H} & (\epsilon' + 3\,\kappa_2\,\epsilon)\,\tilde V_{\overline{\psi}_H} & 0 \\ -(\epsilon' - 3\,\kappa_2\,\epsilon)\,\tilde V_{\overline{\psi}_H} & 3\epsilon\tilde V_{\overline{\psi}_H} & 0 \\ 0 & r\,\epsilon\,(1-\delta)\,T_{\bar\nu}\tilde V_{\overline{\psi}_H} & \tilde V'_{\overline{\psi}_H} \end{pmatrix}, \tag{3.29}$$

$$M_N = \begin{pmatrix} \kappa_1\,\tilde S & \kappa_2\,\tilde S & 0 \\ \kappa_2\,\tilde S & \tilde S & \tilde\phi \\ 0 & \tilde\phi & 0 \end{pmatrix}, \tag{3.30}$$

where $\tilde V_{\overline{\psi}_H}$ and $\tilde V'_{\overline{\psi}_H}$ are proportional to the VEV of $\overline{\psi}_H$ (with different implicit Yukawa couplings), $\tilde S$ and $\tilde\phi$ are up to couplings the VEVs of S^{22} and ϕ^2, respectively.

It has been shown in ref. [12] that the very good fits to the Standard Model fermion (quark, lepton, and neutrino) masses and mixings are obtained from the Yukawa matrices given in eqs. (3.21)–(3.24) and the generalized neutrino mass matrix given in eq. (3.27). In particular, the neutrino mixings can be bilarge, which explain the solar neutrino experiment, the atmospheric neutrino experiment, and the Kamland experiment.

3.2 Model II

In Model II, the zero modes $(\Sigma_i)^{(0)}_{(\mathbf{16},-\mathbf{3})}$ of $(\Sigma_i)_{(\mathbf{16},-\mathbf{3})}$ where $i = 2, 3$ are considered as the first two families of fermions, and the R symmetry $\mathrm{SO}(2)_{56} \times \mathrm{U}(1)_{\mathbf{4}_+} \times \mathrm{SU}(2)_{\mathbf{4}_-}$ can give us the flavour symmetry to generate the fermion masses and mixings.

To be convenient in discussions, we define

$$\mathbf{16}_1 \equiv -(\Sigma_3)^{(0)}_{(\mathbf{16},-\mathbf{3})}, \qquad \mathbf{16}_2 \equiv (\Sigma_2)^{(0)}_{(\mathbf{16},-\mathbf{3})}, \qquad \Upsilon_0 \equiv (\Sigma_1)^{(0)}_{(\mathbf{16},-\mathbf{3})}, \tag{3.31}$$

where $\mathbf{16}_i$ is the $i-th$ family of the Standard Model fermions. In this convention, the first two families $(\mathbf{16}_1, \mathbf{16}_2)^T$ form a fundamental representation under the $\mathrm{SU}(2)_{\mathbf{4}_-}$ R symmetry.

We define the $\mathrm{U}(1)^{II}_R$ and $\mathrm{U}(1)^{II}_F$ global symmetries from the $\mathrm{SO}(2)_{56} \times \mathrm{U}(1)_{\mathbf{4}_+}$ R symmetry, respectively. The generators for $\mathrm{U}(1)^{II}_R$ and $\mathrm{U}(1)^{II}_F$ in $\mathrm{SU}(4)_R$ are

$$T_{\mathrm{U}(1)^{II}_R} = \mathrm{diag}\,[-1, 1, 0, 0]\,, \tag{3.32}$$

$$T_{\mathrm{U}(1)^{II}_F} = \mathrm{diag}\left[\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right]. \tag{3.33}$$

So, under $\mathrm{U}(1)^{II}_R$, the Grassmann coordinate θ has charge $+1$, $\mathbf{16}_1$ and $\mathbf{16}_2$ have charge $+1$, while Υ_0 has charge 0. In addition, under $\mathrm{U}(1)^{II}_F$, the Grassmann coordinate θ has charge $-1/2$, $\mathbf{16}_1$ and $\mathbf{16}_2$ have charge 0, while Υ_0 has charge -1.

On the observable 3-brane at $z = 0$, we introduce the third family of fermion $\mathbf{16}_3$ whose quantum numbers under $\mathrm{U}(1)_X \times \mathrm{U}(1)^{II}_R \times \mathrm{U}(1)^{II}_F$ are $(-3, +1, -1/2)$, and a $\mathbf{16}$ spinor representation Υ'_0 whose quantum numbers under $\mathrm{U}(1)_X \times \mathrm{U}(1)^{II}_R \times \mathrm{U}(1)^{II}_F$ are $(+3, +2, 0)$. So, we can introduce the mass term $M_{\Upsilon_0}\Upsilon_0\Upsilon'_0$ for Υ_0 and Υ'_0 in the superpotential on the 3-brane at $z = 0$. The mass term for Υ_0 and Υ'_0 can also be generated by the superpotential $s_1 s'_1 \Upsilon_0 \Upsilon'_0$ where s_1 and s'_1 will be defined when we discuss the anomaly cancellations.

In Model II, the R symmetry $\mathrm{U}(1)^I_F \times \mathrm{SU}(2)_{\mathbf{4}_-}$ can be considered as flavour symmetry to explain the fermion masses and mixings. Similar to the discussions in Model I, we assume that the particles, which are introduced to break the gauge symmetry and generate the fermion masses and mixings, are on the observable 3-brane at $z = 0$. The additional particle content in Model II is the same as that in Model I. In short, the particles in Model II are the Standard Model fermions $\mathbf{16}_{a'}$, $\mathbf{16}_3$; Higgs fields H, $\mathbf{45}_H$, ψ_H, $\overline{\psi}_H$; vector like particles Υ_0, Υ'_0, ψ, $\overline{\psi}$, $\psi^{a'}$, $\overline{\psi}_{a'}$, $\psi_{a'}$, $\overline{\psi}^{a'}$; three SO(10) singlets $N_{a'}$ and N_3; and Higgs fields for flavour symmetry breaking $\phi^{a'}$, $S^{a'b'}$, and $A^{a'b'}$. The quantum numbers for the particles in Model II under the $\mathrm{U}(1)_X$ gauge symmetry and $\mathrm{U}(1)^{II}_R \times \mathrm{U}(1)^{II}_F$ R symmetry are given in table 2, where Q_X and Q_F are free parameters.

I. Low energy R symmetry. The $\mathrm{U}(1)^{II}_R$ is the traditional $\mathrm{U}(1)_R$ R symmetry defined in the low energy supersymmetry phenomenology. So, the superpartners of the ordinary particles can be the candidates of cold dark matter, for instance, the neutralino.

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Particles	$\mathrm{U}(1)_X \times \mathrm{U}(1)_R^{II} \times \mathrm{U}(1)_F^{II}$
$\mathbf{16}_{a'}$	$(-3;\ +1;\ 0\ )$
$\mathbf{16}_3$	$(-3;\ +1;\ -1/2\ )$
H	$(+6;\ 0;\ 0\ )$
$\mathbf{45}_H$	$(-Q_X;\ 0;\ -Q_F-1)$
ψ_H	$(0;\ 0;\ Q^F_{\psi_H})$
$\overline{\psi}_H$	$(Q_X/2+3;\ 0;\ Q_F/2+1/2\ )$
Υ_0	$(-3;\ 0;\ -1)$
Υ_0'	$(+3;\ 2;\ 0)$
ψ	$(-Q_X-3;\ +1;\ -Q_F-3/2)$
$\overline{\psi}$	$(Q_X+3;\ +1;\ Q_F+1/2)$
$\psi^{a'}$	$(-3;\ +1;\ -1)$
$\overline{\psi}_{a'}$	$(+3;\ +1;\ 0)$
$\psi_{a'}$	$(-Q_X-3;\ +1;\ -Q_F-1)$
$\overline{\psi}^{a'}$	$(Q_X+3;\ +1;\ Q_F)$
$N_{a'}$	$(-Q_X/2;\ +1;\ -Q_F/2-1/2)$
N_3	$(-Q_X/2;\ +1;\ -Q_F/2-1)$
$\phi^{a'}$	$(Q_X;\ 0;\ Q_F+1/2)$
$S^{a'b'}$	$(Q_X;\ 0;\ Q_F)$
$A^{a'b'}$	$(0;\ 0;\ -1)$

Table 2: Model II. Under the $\mathrm{U}(1)_X$ gauge symmetry and $\mathrm{U}(1)_R^{II} \times \mathrm{U}(1)_F^{II}$ R symmetry, the quantum numbers for the Standard Model fermions $\mathbf{16}_{a'}$, $\mathbf{16}_3$; Higgs fields H, $\mathbf{45}_H$, ψ_H, $\overline{\psi}_H$; vector like particles Υ_0, Υ_0', ψ, $\overline{\psi}$, $\psi^{a'}$, $\overline{\psi}_{a'}$, $\psi_{a'}$, $\overline{\psi}^{a'}$; three SO(10) singlets $N_{a'}$ and N_3; and Higgs fields for flavour symmetry breaking $\phi^{a'}$, $S^{a'b'}$, and $A^{a'b'}$. In this table, Q_X and Q_F are free parameters. And the $\mathrm{U}(1)_F^{II}$ charge for ψ_H ($Q^F_{\psi_H}$) is arbitrary because we have no constraints from superpotential.

II. Anomaly cancellations. Similar to the discussions in the subsection 3.1, the anomalies $[\mathrm{SO}(10)]^2\,\mathrm{U}(1)_X$, $[\mathrm{U}(1)_X]^3$, and $[\mathrm{Gravity}]^2\,\mathrm{U}(1)_X$ can be cancelled by a generalized Green-Schwarz mechanism. Here, we discuss the anomaly cancellations by introducing the vector-like particles on three 3-branes at three fixed points.

Similar to the anomaly discussions in Model I, the anomalies localized on the 3-brane at each fixed point due to the bulk vector multiplet are equivalent to the anomalies from a multiplet with quantum number $(\mathbf{16}, \mathbf{-3})$ under the $\mathrm{SO}(10) \times \mathrm{U}(1)_X$ gauge symmetry. If we consider the anomaly inflows [37, 38], like the 5-dimensional orbifold theories, the 4-dimensional anomaly cancellations are sufficient to ensure the consistency of higher dimensional orbifold theories [37]. Thus, we only need to consider the total anomalies due to the bulk zero modes and the particles localized on the 3-branes at the fixed points.

To be concrete, we still discuss the anomaly cancellations on the 3-brane at each fixed point, which are similar to those in Model I. For simplicity, we consider the case with $Q_X = 0$.

First, on the observable 3-brane at $z = 0$, the anomalies due to the zero modes of bulk vector multiplet are cancelled by the Higgs field $\overline{\psi}_H$. And to cancel the anomalies generated by the Higgs field H, we introduce one pair of 10-plets T_{10} and T'_{10} whose quantum numbers are $(\mathbf{10}, -\mathbf{6})$ and $(\mathbf{10}, \mathbf{0})$ respectively under the $\mathrm{SO}(10) \times \mathrm{U}(1)_X$ gauge symmetry. In order to give the masses to T_{10} and T'_{10}, and keep the $\mathrm{U}(1)_X$ D-flatness and anomaly free on the 3-brane at $z = 0$, we introduce the SO(10) singlets s_1 and s'_1 with $\mathrm{U}(1)_X$ charges $+6$ and -6, respectively. Moreover, to forbid the mixings between the Higgs H and T_{10} or between H and T'_{10}, we assign the following $\mathrm{U}(1)_R^{II} \times \mathrm{U}(1)_F^{II}$ charges for the vector-like particles: the $\mathrm{U}(1)_R^{II}$ charges for T_{10}, T'_{10}, s_1 and s'_1 are 1, 1, 0 and 0, respectively, and the $\mathrm{U}(1)_F^{II}$ charges for T_{10}, T'_{10}, s_1 and s'_1 are $-1/2$, $-1/2$, 0 and 0, respectively. In short, the anomalies $[\mathrm{SO}(10)]^2\,\mathrm{U}(1)_X$, $[\mathrm{U}(1)_X]^3$, and $[\mathrm{Gravity}]^2\,\mathrm{U}(1)_X$ are cancelled on the 3-brane at $z = 0$. And the masses of T_{10} and T'_{10} can be generated by the superpotential $s_1 T_{10} T'_{10}$.

Second, on the 3-brane at $z = \pi R e^{\mathrm{i}\pi/6}/\sqrt{3}$, we need to cancel the anomalies generated by a multiplet with quantum number $(\mathbf{16}, -\mathbf{3})$ under the $\mathrm{SO}(10) \times \mathrm{U}(1)_X$ gauge symmetry. So, we introduce a pair of the Υ_2 and Υ'_2 whose quantum numbers are $(\mathbf{16}, \mathbf{0})$ and $(\overline{\mathbf{16}}, +\mathbf{3})$ respectively under the $\mathrm{SO}(10) \times \mathrm{U}(1)_X$ gauge symmetry. And in order to give the masses to Υ_2 and Υ'_2, and keep the $\mathrm{U}(1)_X$ D-flatness and anomaly free on the 3-brane at $z = \pi R e^{\mathrm{i}\pi/6}/\sqrt{3}$, we introduce the SO(10) singlets s_2 and s'_2 with $\mathrm{U}(1)_X$ charges -3 and $+3$, respectively. In addition, to forbid the mixings between the SM fermions $\mathbf{16}_j$ ($j = 1, 2$) and Υ_2, and the mass terms between $\mathbf{16}_j$ and Υ'_2, we assign the following $\mathrm{U}(1)_R^{II} \times \mathrm{U}(1)_F^{II}$ charges for the vector-like particles: the $\mathrm{U}(1)_R^{II}$ charges for Υ_2, Υ'_2, s_2 and s'_2 are 1, 1, 0 and 0, respectively, and the $\mathrm{U}(1)_F^{II}$ charges for Υ_2, Υ'_2, s_2 and s'_2 are $-1/2$, $-1/2$, 0 and 0, respectively. And the masses of Υ_2 and Υ'_2 can be generated by the superpotential $s_2 \Upsilon_2 \Upsilon'_2$.

Third, the anomaly cancellations on the 3-brane at $z = 2\pi R e^{\mathrm{i}\pi/6}/\sqrt{3}$ is similar to those on the 3-brane at $z = \pi R e^{\mathrm{i}\pi/6}/\sqrt{3}$. We introduce the particles Υ_3, Υ'_3, s_3 and s'_3, whose quantum numbers are the same as those for Υ_2, Υ'_2, s_2 and s'_2, respectively.

III. Fermion masses and mixings. With the particle quantum numbers given in table 2, we obtain that the superpotential in Model II is the same as that in Model I given by eq. (3.18). Therefore, the discussions for the fermion masses and mixings are the same as those in Model I, too. The effective Yukawa matrices for up-type quarks, down-type quarks, and charged leptons, and the Dirac neutrino Yukawa matrix are given by eqs. (3.21)–(3.24), and the generalized neutrino mass matrix is given by eq. (3.27).

In addition, we consider the cut-off scale (M_*) is close to the compactification scale ($1/R$) due to the unitarity constraints [39]. Then, comparing to the Yukawa couplings of the third family, the volume suppression factor $1/(M_* R)$ for the Yukawa couplings of the first two families of fermions is close to 1, *i.e*, at the order of 1 ($\mathcal{O}(1)$).

In short, we can also have very good fits to the Standard Model fermion masses and mixings, especially the neutrino mixings can be bilarge.

4. Comments on the $\mathcal{N}=(1,1)$ supersymmetric E_6 models on $M^4 \times T^2/Z_4$ and $M^4 \times T^2/Z_6$

We are only interested in the models where there are at least two families of the SM fermions from the zero modes of bulk vector multiplet. In this section, we comment on the $\mathcal{N}=(1,1)$ supersymmetric E_6 models on $M^4 \times T^2/Z_n$ in which $n=4$ and 6.

The discussions for E_6 symmetry breaking are similar to those in the section 3. The $R_\omega^{E_6}$ for E_6 gauge group is the product of the $R_\omega^{\mathrm{SO}(10)}$ for SO(10) and $R_\omega^{\mathrm{U}(1)_X}$ for $\mathrm{U}(1)_X$, and we choose

$$R_\omega^{\mathrm{SO}(10)} = \mathrm{diag}[+1,+1,+1,+1,+1,+1,+1,+1,+1,+1], \tag{4.1}$$

$$R_\omega^{\mathrm{U}(1)_X} = \exp\left(-\mathrm{i}\frac{2\pi Q}{3n}\right), \tag{4.2}$$

where $n=4$ and 6 for the T^2/Z_4 and T^2/Z_6 orbifolds, respectively.

In order to have two families of the SM fermions from the zero modes of bulk vector multiplet, we obtain $k=0$. Using eqs. (2.43)–(2.46) and (4.1)–(4.2), we obtain the transformations of vector multiplet

$$V_{(\mathbf{45},\mathbf{0})}(x^\mu,\, \omega z,\, \omega^{-1}\bar{z}) = V_{(\mathbf{45},\mathbf{0})}(x^\mu,\, z,\, \bar{z})\,, \tag{4.3}$$

$$V_{(\mathbf{1},\mathbf{0})}(x^\mu,\, \omega z,\, \omega^{-1}\bar{z}) = V_{(\mathbf{1},\mathbf{0})}(x^\mu,\, z,\, \bar{z})\,, \tag{4.4}$$

$$V_{(\mathbf{16},-\mathbf{3})}(x^\mu,\, \omega z,\, \omega^{-1}\bar{z}) = \omega V_{(\mathbf{16},-\mathbf{3})}(x^\mu,\, z,\, \bar{z})\,, \tag{4.5}$$

$$V_{(\overline{\mathbf{16}},\mathbf{3})}(x^\mu,\, \omega z,\, \omega^{-1}\bar{z}) = \omega^{n-1} V_{(\overline{\mathbf{16}},\mathbf{3})}(x^\mu,\, z,\, \bar{z})\,, \tag{4.6}$$

$$(\Sigma_j)_{(\mathbf{45},\mathbf{0})}(x^\mu,\, \omega z,\, \omega^{-1}\bar{z}) = \omega^{n-1} (\Sigma_j)_{(\mathbf{45},\mathbf{0})}(x^\mu,\, z,\, \bar{z})\,, \tag{4.7}$$

$$(\Sigma_j)_{(\mathbf{1},\mathbf{0})}(x^\mu,\, \omega z,\, \omega^{-1}\bar{z}) = \omega^{n-1} (\Sigma_j)_{(\mathbf{1},\mathbf{0})}(x^\mu,\, z,\, \bar{z})\,, \tag{4.8}$$

$$(\Sigma_j)_{(\mathbf{16},-\mathbf{3})}(x^\mu,\, \omega z,\, \omega^{-1}\bar{z}) = (\Sigma_j)_{(\mathbf{16},-\mathbf{3})}(x^\mu,\, z,\, \bar{z})\,, \tag{4.9}$$

$$(\Sigma_j)_{(\overline{\mathbf{16}},\mathbf{3})}(x^\mu,\, \omega z,\, \omega^{-1}\bar{z}) = \omega^{n-2} (\Sigma_j)_{(\overline{\mathbf{16}},\mathbf{3})}(x^\mu,\, z,\, \bar{z})\,, \tag{4.10}$$

$$(\Sigma_3)_{(\mathbf{45},\mathbf{0})}(x^\mu,\, \omega z,\, \omega^{-1}\bar{z}) = \omega^2 (\Sigma_3)_{(\mathbf{45},\mathbf{0})}(x^\mu,\, z,\, \bar{z})\,, \tag{4.11}$$

$$(\Sigma_3)_{(\mathbf{1},\mathbf{0})}(x^\mu,\, \omega z,\, \omega^{-1}\bar{z}) = \omega^2 (\Sigma_3)_{(\mathbf{1},\mathbf{0})}(x^\mu,\, z,\, \bar{z})\,, \tag{4.12}$$

$$(\Sigma_3)_{(\mathbf{16},-\mathbf{3})}(x^\mu,\, \omega z,\, \omega^{-1}\bar{z}) = \omega^3 (\Sigma_3)_{(\mathbf{16},-\mathbf{3})}(x^\mu,\, z,\, \bar{z})\,, \tag{4.13}$$

$$(\Sigma_3)_{(\overline{\mathbf{16}},\mathbf{3})}(x^\mu,\, \omega z,\, \omega^{-1}\bar{z}) = \omega (\Sigma_3)_{(\overline{\mathbf{16}},\mathbf{3})}(x^\mu,\, z,\, \bar{z})\,, \tag{4.14}$$

where $j=1,2$, and $\omega = e^{\mathrm{i}2\pi/n}$ $(n=4,6)$.

From the general discussions on gauge symmetry breaking by discrete symmetry [26], we obtain that only the fields $V_{(\mathbf{45},\mathbf{0})}$, $V_{(\mathbf{1},\mathbf{0})}$, $(\Sigma_1)_{(\mathbf{16},-\mathbf{3})}$ and $(\Sigma_2)_{(\mathbf{16},-\mathbf{3})}$ have zero modes. So, the E_6 gauge symmetry is broken down to $\mathrm{SO}(10)\times\mathrm{U}(1)_X$ for the zero modes by orbifold projection. And there are two SO(10) spinor representations **16** from the zero modes of bulk vector multiplet, which can be considered as two families of fermions. The R symmetry is $\mathrm{SO}(2)_{56}\times\mathrm{U}(1)_{\mathbf{4}_+}\times\mathrm{U}(1)_{\mathbf{4}_-}$, which can be the origin of flavour symmetry.

We define the zero modes $(\Sigma_1)^{(0)}_{(\mathbf{16},-\mathbf{3})}$ and $(\Sigma_2)^{(0)}_{(\mathbf{16},-\mathbf{3})}$ of $(\Sigma_1)_{(\mathbf{16},-\mathbf{3})}$ and $(\Sigma_2)_{(\mathbf{16},-\mathbf{3})}$ as the first two families of fermions

$$\mathbf{16}_1 \equiv (\Sigma_1)^{(0)}_{(\mathbf{16},-\mathbf{3})}\,, \qquad \mathbf{16}_2 \equiv (\Sigma_2)^{(0)}_{(\mathbf{16},-\mathbf{3})}\,. \tag{4.15}$$

To be convenient, we define three $\mathrm{U}(1)_R^{III}$, $\mathrm{U}(1)_F^{III}$ and $\mathrm{U}(1)_{RF}$ global symmetries from the R symmetry $\mathrm{SO}(2)_{56} \times \mathrm{U}(1)_{\mathbf{4}_+} \times \mathrm{U}(1)_{\mathbf{4}_-}$, whose generators in $\mathrm{SU}(4)_R$ are

$$T_{\mathrm{U}(1)_R^{III}} = \mathrm{diag}\,[-1, 0, 0, +1]\,, \tag{4.16}$$

$$T_{\mathrm{U}(1)_F^{III}} = \mathrm{diag}\left[0, \frac{1}{2}, -\frac{1}{2}, 0\right], \tag{4.17}$$

$$T_{\mathrm{U}(1)_{RF}} = \mathrm{diag}\left[\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right]. \tag{4.18}$$

Under $\mathrm{U}(1)_R^{III}$ symmetry, the Grassmann coordinate θ, $\mathbf{16}_1$ and $\mathbf{16}_2$ have charge $+1$, so, the $\mathrm{U}(1)_R^{III}$ is the traditional $\mathrm{U}(1)_R$ R symmetry in the low energy supersymmetry phenomenology. And under $\mathrm{U}(1)_F^{III}$ global symmetry, the Grassmann coordinate θ has charge 0, $\mathbf{16}_1$ and $\mathbf{16}_2$ have charges $-1/2$ and $+1/2$, respectively. In addition, under $\mathrm{U}(1)_{RF}$ global symmetry, the Grassmann coordinate θ has charge $-1/2$, and the $\mathbf{16}_1$ and $\mathbf{16}_2$ have charge 0. The $\mathrm{U}(1)_F^{III} \times \mathrm{U}(1)_{RF}$ R symmetry can be considered as flavour symmetry.

Similar to the discussions in section 3, the anomalies $[\mathrm{SO}(10)]^2\,\mathrm{U}(1)_X$, $[\mathrm{U}(1)_X]^3$, and $[\mathrm{Gravity}]^2\,\mathrm{U}(1)_X$, can be cancelled by a generalized Green-Schwarz mechanism, or by adding the suitable vector-like particles on the 3-branes at the fixed points. If we introduce the suitable vector-like particles on the 3-branes at the fixed points to cancel the anomalies, it is easier if we consider the anomaly inflows among the 3-branes at the fixed points.

The supersymmetric SO(10) models with U(1) and extra discrete flavour symmetry [9, 10] or $\mathrm{U}(1)_A$ symmetry [19] have been studied previously, and those discussions on fermion masses and mixings may be generalized to these models. However, how to explain the fermion masses and mixings in the supersymmetric $\mathrm{SO}(10) \times \mathrm{U}(1)_X$ models with $\mathrm{U}(1)_F^{III} \times \mathrm{U}(1)_{RF}$ flavour symmetry is still an interesting question and deserves further detail study.

5. Comments on the $\mathcal{N} = (1, 1)$ supersymmetric SU(9) and SU(8) models on T^2 orbifolds

In this section, we briefly discuss the SU(9) models on the space-time $M^4 \times T^2/Z_3$ where there are three or two families of fermions from the zero modes of bulk vector multiplet. We also briefly discuss the SU(8) Models on the space-times $M^4 \times T^2/Z_3$, $M^4 \times T^2/Z_4$ and $M^4 \times T^2/Z_6$ where there are three or two families from the zero modes of bulk vector multiplet.

5.1 The SU(9) models on $M^4 \times T^2/Z_3$

These models have been briefly commented in ref. [33]. In order to break the SU(9) gauge symmetry, we choose R_ω in the adjoint representation of SU(9)

$$R_\omega^{\mathrm{SU}(9)} = \mathrm{diag}[+1, +1, +1, \omega^2, \omega^2, \omega^2, \omega, \omega, \omega]\,, \tag{5.1}$$

where $\omega = e^{\mathrm{i}2\pi/3}$.

Under $R_\omega^{\mathrm{SU}(9)}$, the SU(9) gauge symmetry is broken down to the $\mathrm{SU}(3)_C \times \mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1) \times \mathrm{U}(1)'$ gauge symmetry. And the generators for U(1) and U(1)′ are

$$T_{\mathrm{U}(1)} = \mathrm{diag}\,[+1,+1,+1,-1,-1,-1,0,0,0]\,, \tag{5.2}$$

$$T_{\mathrm{U}(1)'} = \mathrm{diag}\,[+1,+1,+1,+1,+1,+1,-2,-2,-2]\,. \tag{5.3}$$

Under the $\mathrm{SU}(3)_C \times \mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1) \times \mathrm{U}(1)'$ gauge symmetry, the SU(9) adjoint representation **80** is decomposed as

$$\mathbf{80} = MG(\mathbf{78}) + (\mathbf{1},\mathbf{1},\mathbf{1})_{(0,0)} + (\mathbf{1},\mathbf{1},\mathbf{1})_{(0,0)}\,, \tag{5.4}$$

where

$$\mathbf{MG}(\mathbf{78}) = \begin{pmatrix} (\mathbf{8},\mathbf{1},\mathbf{1})_{(0,0)} & (\mathbf{3},\bar{\mathbf{3}},\mathbf{1})_{(2,0)} & (\mathbf{3},\mathbf{1},\bar{\mathbf{3}})_{(1,3)} \\ (\bar{\mathbf{3}},\mathbf{3},\mathbf{1})_{(-2,0)} & (\mathbf{1},\mathbf{8},\mathbf{1})_{(0,0)} & (\mathbf{1},\mathbf{3},\bar{\mathbf{3}})_{(-1,3)} \\ (\bar{\mathbf{3}},\mathbf{1},\mathbf{3})_{(-1,-3)} & (\mathbf{1},\bar{\mathbf{3}},\mathbf{3})_{(1,-3)} & (\mathbf{1},\mathbf{1},\mathbf{8})_{(0,0)} \end{pmatrix}, \tag{5.5}$$

where the subscripts denote the charges under the $\mathrm{U}(1) \times \mathrm{U}(1)'$ gauge symmetry.

Note that $k = 0$ for T^2/Z_3 orbifold. Using eqs. (2.43)–(2.46) and (5.1), we obtain that the Z_3 transformation properties for these decomposed representations of the vector multiplet V and chiral multiplets Σ_i are

$$V:\ \begin{pmatrix} 1 & \omega & \omega^2 \\ \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \end{pmatrix} + (1) + (1)\,, \qquad \Sigma_i:\ \begin{pmatrix} \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \\ 1 & \omega & \omega^2 \end{pmatrix} + (\omega^2) + (\omega^2)\,. \tag{5.6}$$

So, the $(\mathbf{3},\bar{\mathbf{3}},\mathbf{1})_{(2,0)}$, $(\mathbf{1},\mathbf{3},\bar{\mathbf{3}})_{(-1,3)}$, and $(\bar{\mathbf{3}},\mathbf{1},\mathbf{3})_{(-1,-3)}$ in Σ_i $(i = 1,2,3)$ have the zero modes. And these zero modes can be considered as three families or two families of fermions in the "trinification".

Similar to the discussions in section 3, we can obtain the SU(9) model with $\mathrm{U}(1)_F^I \times \mathrm{SU}(2)_{\mathbf{4}_-}$ flavour symmetry and three families of fermions from the zero modes of bulk vector multiplet, and the SU(9) model with $\mathrm{U}(1)_F^{II} \times \mathrm{SU}(2)_{\mathbf{4}_-}$ flavour symmetry and two families from the zero modes of bulk vector multiplet. Therefore, how to explain the fermion masses and mixings in the extra "trinification" $\mathrm{SU}(3)_C \times \mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1) \times \mathrm{U}(1)'$ with these flavour symmetries is another interesting question. By the way, the anomalies due to the $\mathrm{U}(1) \times \mathrm{U}(1)'$ gauge symmetry can be cancelled similarly.

5.2 The SU(8) models on $M^4 \times T^2/Z_n$

The SU(8) model on $M^4 \times T^2/Z_6$ has been discussed previously where the electroweak (EW) breaking Higgs fields as well as the third family of fermions are unified into the 6-dimensional vector multiplet [28]. Here, we concentrate on the models with three or two families of fermions from the zero modes of bulk vector multiplet.

I. SU(8) models on $M^4 \times T^2/Z_3$. In order to break the SU(8) gauge symmetry, we choose R_ω in the adjoint representation of SU(8)

$$R_\omega^{\mathrm{SU}(8)} = \mathrm{diag}[+1,+1,+1,+1,\omega^2,\omega^2,\omega,\omega]\,, \tag{5.7}$$

where $\omega = e^{\mathrm{i}2\pi/3}$.

Under $R_\omega^{\mathrm{SU}(8)}$, the SU(8) gauge symmetry is broken down to the $\mathrm{SU}(4)_C \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1) \times \mathrm{U}(1)'$ gauge symmetry. And the generators for U(1) and U(1)$'$ are

$$T_{\mathrm{U}(1)} = \mathrm{diag}\,[+1,+1,+1,+1,-1,-1,-1,-1]\,, \tag{5.8}$$
$$T_{\mathrm{U}(1)'} = \mathrm{diag}\,[0,0,0,0,+1,+1,-1,-1]\,. \tag{5.9}$$

Under the $\mathrm{SU}(4)_C \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1) \times \mathrm{U}(1)'$ gauge symmetry, the SU(8) adjoint representation **63** is decomposed as

$$\mathbf{63} = MG(\mathbf{61}) + (\mathbf{1},\mathbf{1},\mathbf{1})_{(0,0)} + (\mathbf{1},\mathbf{1},\mathbf{1})_{(0,0)}\,, \tag{5.10}$$

where

$$\mathbf{MG}(\mathbf{61}) = \begin{pmatrix} (\mathbf{15},\mathbf{1},\mathbf{1})_{(0,0)} & (\mathbf{4},\mathbf{2},\mathbf{1})_{(2,-1)} & (\mathbf{4},\mathbf{1},\mathbf{2})_{(2,1)} \\ (\bar{\mathbf{4}},\mathbf{2},\mathbf{1})_{(-2,1)} & (\mathbf{1},\mathbf{3},\mathbf{1})_{(0,0)} & (\mathbf{1},\mathbf{2},\mathbf{2})_{(0,2)} \\ (\bar{\mathbf{4}},\mathbf{1},\mathbf{2})_{(-2,-1)} & (\mathbf{1},\mathbf{2},\mathbf{2})_{(0,-2)} & (\mathbf{1},\mathbf{1},\mathbf{3})_{(0,0)} \end{pmatrix}, \tag{5.11}$$

where the subscripts denote the charges under the $\mathrm{U}(1) \times \mathrm{U}(1)'$ gauge symmetry.

Note that $k = 0$ for T^2/Z_3 orbifold. Using eqs. (2.43)–(2.46) and (5.7), we obtain that the Z_3 transformation properties for these decomposed representations of the vector multiplet V and chiral multiplets Σ_i are

$$V:\ \begin{pmatrix} 1 & \omega & \omega^2 \\ \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \end{pmatrix} + (1) + (1)\,, \qquad \Sigma_i:\ \begin{pmatrix} \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \\ 1 & \omega & \omega^2 \end{pmatrix} + (\omega^2) + (\omega^2)\,. \tag{5.12}$$

So, the $(\mathbf{4},\mathbf{2},\mathbf{1})_{(2,-1)}$, $(\bar{\mathbf{4}},\mathbf{1},\mathbf{2})_{(-2,-1)}$ and $(\mathbf{1},\mathbf{2},\mathbf{2})_{(0,2)}$ in Σ_i ($i = 1,2,3$) have the zero modes. The zero modes of $(\mathbf{4},\mathbf{2},\mathbf{1})_{(2,-1)}$ and $(\bar{\mathbf{4}},\mathbf{1},\mathbf{2})_{(-2,-1)}$ in Σ_i can be considered as three families of the SM fermions in the Pati-Salam model, while the $(\mathbf{1},\mathbf{2},\mathbf{2})_{(0,2)}$ in Σ_i can be considered as three bi-doublet EW breaking Higgs particles. Unfortunately, these Higgs fields can not give the correct masses and mixings for the Standard Model fermions unless we add various Yukawa terms on the 3-branes at the fixed points, so, we do not consider them as the EW breaking Higgs particles, i.e., we give up the gauge-Higgs-fermion unification or the gauge-Yukawa unification. For simplicity, we introduce the Standard Model EW breaking Higgs fields on the observable 3-brane at the fixed point $z = 0$, which are similar to those in section 3.

Similar to the discussions in section 3, we can have the SU(8) model with $\mathrm{U}(1)_F^I \times \mathrm{SU}(2)_{\mathbf{4}_-}$ flavour symmetry and three families of fermions from the zero modes of bulk vector multiplet, and the SU(8) model with $\mathrm{U}(1)_F^{II} \times \mathrm{SU}(2)_{\mathbf{4}_-}$ flavour symmetry and two families of fermions from the zero modes of bulk vector multiplet. Essentially speaking, the SO(10) models are similar to the Pati-Salam models when we consider the fermion masses and mixings. Thus, similar to the discussions in section 3, we can explain the fermion masses and mixings via these flavour symmetries. And three bi-doublets from the zero modes of bulk vector multiplet can obtain the heavy masses if we couple them with the singlets which have non-zero VEVs on the 3-branes at the fixed points. By the way, the discussions for anomaly cancellations due to the $\mathrm{U}(1) \times \mathrm{U}(1)'$ gauge symmetry are similar to those in section 3, too.

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II. SU(8) **models on** $M^4 \times T^2/Z_4$ **and** $M^4 \times T^2/Z_6$**.** We consider the SU(8) Models on the space-time $M^4 \times T^2/Z_n$ where $n = 4, 6$. In order to break the SU(8) gauge symmetry, we choose

$$R_\omega^{\mathrm{SU}(8)} = \mathrm{diag}[+1, +1, +1, +1, \omega^{n-1}, \omega^{n-1}, \omega, \omega]\,, \tag{5.13}$$

where $\omega = e^{\mathrm{i}2\pi/n}$ with $n = 4, 6$. Under $R_\omega^{\mathrm{SU}(8)}$, the SU(8) gauge symmetry is broken down to the $\mathrm{SU}(4)_C \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1) \times \mathrm{U}(1)'$ gauge symmetry.

Using eqs. (2.43)–(2.46) with $k = 0$ and eq. (5.13), we obtain that the Z_n transformation properties for these decomposed representations of the vector multiplet V and chiral multiplets Σ_i are

$$V:\ \begin{pmatrix} 1 & \omega & \omega^{n-1} \\ \omega^{n-1} & 1 & \omega^{n-2} \\ \omega & \omega^2 & 1 \end{pmatrix} + (1) + (1)\,, \qquad \Sigma_3:\ \begin{pmatrix} \omega^2 & \omega^3 & \omega \\ \omega & \omega^2 & 1 \\ \omega^3 & \omega^4 & \omega^2 \end{pmatrix} + (\omega^2) + (\omega^2)\,, \tag{5.14}$$

$$\Sigma_j:\ \begin{pmatrix} \omega^{n-1} & 1 & \omega^{n-2} \\ \omega^{n-2} & \omega^{n-1} & \omega^{n-3} \\ 1 & \omega & \omega^{n-1} \end{pmatrix} + (\omega^{n-1}) + (\omega^{n-1})\,, \tag{5.15}$$

where $j = 1, 2$. So, the $(\mathbf{4}, \mathbf{2}, \mathbf{1})_{(2,-1)}$ and $(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2})_{(-2,-1)}$ in Σ_j ($j = 1, 2$) have zero modes, which can be considered as two families of fermions in the Pati-Salam model. The $(\mathbf{1}, \mathbf{2}, \mathbf{2})_{(0,2)}$ and $(\mathbf{1}, \mathbf{2}, \mathbf{2})_{(0,-2)}$ in Σ_3 for T^2/Z_4 orbifold, and the $(\mathbf{1}, \mathbf{2}, \mathbf{2})_{(0,2)}$ in Σ_3 for T^2/Z_6 orbifold have zero modes, too. The $(\mathbf{1}, \mathbf{2}, \mathbf{2})_{(0,2)}$ in Σ_3 can be considered as a bi-doublet Higgs particle. Similar to the SU(8) models on $M^4 \times T^2/Z_3$, this bi-doublet Higgs particle can not give the correct Standard Model fermion masses and mixings unless we add various Yukawa terms on the 3-branes at the fixed points, so, we do not consider it as the EW breaking Higgs particle.

Similar to the discussions in section 4, we can have $\mathrm{U}(1)_F^{III} \times \mathrm{U}(1)_{RF}$ R symmetry which can be considered as flavour symmetry. Because the SO(10) models are similar to the Pati-Salam models when we consider the fermion masses and mixings, like the models in section 4, how to explain the fermion masses and mixings by this flavour symmetry deserves further study in detail.

6. Discussions and conclusions

The other interesting flavour symmetry is SU(3). There are two possibilities for SU(3) flavour symmetry: one is the gauged flavour symmetry, the other is the global flavour symmetry. For the gauged SU(3) flavour symmetry, one can consider the GUT group with rank 7 or higher, for example E_8 [29]. And for the global SU(3) flavour symmetry, we should consider the 10-dimensional space-time if this SU(3) flavour symmetry arises from the R symmetry. In the 10-dimensional $\mathcal{N} = 1$ supersymmetry, the SO(6) ($\mathrm{SO}(9,1) \longrightarrow \mathrm{SO}(3,1) \times \mathrm{SO}(6)$) rotation on the extra 6-dimensional space manifold becomes the R symmetry after compactification. Note that the Lie algebra of SO(6) is isomorphic to that of SU(4) and we only keep the 4-dimensional $\mathcal{N} = 1$ supersymmetry after compactification, similar to the discussions in sections 2 and 3, the $\mathrm{SU}(4)_R/\,\mathrm{U}(1)_R$ or

$\mathrm{SU}(3)_R$ R symmetry may become the flavour symmetry if we consider the T^6/Z_3 orbifold. However, the 10-dimensional $\mathcal{N}=1$ supersymmetric non-abelian gauge theory generically has gauge anomaly, and can be cancelled if and only if the gauge groups are SO(32), $E_8 \times E_8$, $E_8 \times \mathrm{U}(1)^{248}$ and $\mathrm{U}(1)^{496}$. Thus, it is difficult to construct the GUT models with global SU(3) flavour symmetry from R symmetry because it is not easy to break these GUT groups down to the Standard Model gauge group.

In this paper, we discuss the 6-dimensional $\mathcal{N}=(1,1)$ supersymmetric gauge theory on the space-time $M^4 \times T^2$, where the R symmetry is $\mathrm{SO}(2)_{56} \times \mathrm{SU}(2)_{\mathbf{4}_+} \times \mathrm{SU}(2)_{\mathbf{4}_-}$. In order to preserve only the 4-dimensional $\mathcal{N}=1$ supersymmetry, we consider the supersymmetry breaking and R symmetry breaking on the T^2/Z_n orbifolds. We find that for T^2/Z_3 orbifold, the only possibility is $k=0$ and the R symmetry is $\mathrm{SO}(2)_{56} \times \mathrm{U}(1)_{\mathbf{4}_+} \times \mathrm{SU}(2)_{\mathbf{4}_-}$. For T^2/Z_4 orbifold, the only possibility is $k=0$, too. And for T^2/Z_6 orbifold, there are two possibilities: $k=0$ and $k=1$. The R symmetry for the T^2/Z_4 and T^2/Z_6 orbifolds is $\mathrm{SO}(2)_{56} \times \mathrm{U}(1)_{\mathbf{4}_+} \times \mathrm{U}(1)_{\mathbf{4}_-}$. We also present the transformation properties of the bulk vector multiplet under these R symmetries.

In the 6-dimensional $\mathcal{N}=(1,1)$ supersymmetric gauge theory, the Standard Model fermions may come from the zero modes of bulk vector multiplet due to the orbifold gauge symmetry breaking, which may explain why there are three families of fermions in the Standard Model. And the R symmetry can give us the flavour symmetry to explain the fermion masses and mixings. To be concrete, we discuss the $\mathcal{N}=(1,1)$ supersymmetric E_6 models on the space-time $M^4 \times T^2/Z_3$ where the gauge symmetry E_6 is broken down to $\mathrm{SO}(10) \times \mathrm{U}(1)_X$ by orbifold projection. There are three SO(10) spinor representation **16**s from the zero modes of bulk vector multiplet, which can be considered as three families of fermions. We study two models in detail. In Model I, three families of the Standard Model fermions arise from the zero modes of bulk vector multiplet, and the R symmetry $\mathrm{U}(1)_F^I \times \mathrm{SU}(2)_{\mathbf{4}_-}$ can be considered as flavour symmetry. However, the $\mathrm{U}(1)_R^I$ R symmetry is not the traditional $\mathrm{U}(1)_R$ R symmetry in the low energy supersymmetry phenomenology, and then, there are no cold dark matter candidates (for example neutralino) from the superpartners. In Model II, the first two families of the Standard Model fermions arise from the zero modes of bulk vector multiplet, and the R symmetry $\mathrm{U}(1)_F^{II} \times \mathrm{SU}(2)_{\mathbf{4}_-}$ can be considered as flavour symmetry. The $\mathrm{U}(1)_R^{II}$ R symmetry is the traditional $\mathrm{U}(1)_R$ R symmetry in the low energy supersymmetry phenomenology. In these models, the anomalies can be cancelled, and we have very good fits to the Standard Model fermion masses and mixings.

Furthermore, we comment on the $\mathcal{N}=(1,1)$ supersymmetric E_6 models on $M^4 \times T^2/Z_4$ and $M^4 \times T^2/Z_6$ where there are two families of the Standard Model fermions from the zero modes of bulk vector multiplet. We also comment on the $\mathcal{N}=(1,1)$ supersymmetric SU(9) models on $M^4 \times T^2/Z_3$, and the SU(8) models on $M^4 \times T^2/Z_n$ orbifolds in which $n=3,4,6$.

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